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ANALOGOUS CHARACTERIZATIONS OF FINITE AND ISOLATED SETS

J. BARBACK, W. D. JACKSON, and M. PARNES

Introduction. Let $E = \{0, 1, 2, \ldots\}$. Members of E will be called *numbers*. A *set* shall mean a subset of E, and a *function* shall mean a function from a set into E. For a function f, then δf will denote its domain. Post [2] introduced simple sets; i.e., recursively enumberable (r.e.) sets with infinite isolated complements. Dekker [1] observed that if Dedekind's definition of finiteness (α is finite iff α is not equivalent to any proper subset of α) is made effective in a natural way, then exactly the class of isolated sets is obtained. The purpose of this note is to characterize when a set α is finite, by giving a condition $\mathrm{C}(\alpha)$ that involves partial orderings of α, and proving

Theorem A. *α is finite* $\Longleftrightarrow \mathrm{C}(\alpha)$.

In addition, we effectivize, in the spirit of Dekker, the condition $\mathrm{C}(\alpha)$ obtaining $\mathrm{C}^e(\alpha)$, and prove

Theorem B. *α is isolated* $\Longleftrightarrow \mathrm{C}^e(\alpha)$.

1. $\mathrm{C}(\alpha)$ *and* $\mathrm{C}^e(\alpha)$. We write P_α if α is a set and P is a binary relation that partially orders α. If P_α and Q_β, then we write $P_\alpha \leq Q_\beta$ if there is a function f such that

$$(1)\qquad \begin{cases} \alpha \subseteq \delta f,\ f \text{ is one-to-one on } \alpha, f(\alpha) \subseteq \beta, \\ \text{and } (\forall x, y \in \alpha)\ [xPy \Rightarrow f(x)\, Qf(y)]. \end{cases}$$

The condition $\mathrm{C}(\alpha)$ is defined by,

$$(\forall P, Q)\ [P_\alpha \leq Q_\alpha \text{ and } Q_\alpha \leq P_\alpha \Rightarrow P_\alpha \sim Q_\alpha],$$

where $P_\alpha \sim Q_\alpha$ means that there is a function f such that,

$$(2)\qquad \begin{cases} \alpha \subseteq \delta f, f \text{ is one-to-one on } \alpha, f(\alpha) = \alpha \\ \text{and } (\forall x, y \in \alpha)\ [xPy \Longleftrightarrow f(x)\, Qf(y)]. \end{cases}$$

If P_α and Q_β then we write $P_\alpha \leq^* Q_\beta$ if there is a partial recursive function f that satisfies (1). The condition $\mathrm{C}^e(\alpha)$ is defined by

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