Notre Dame Journal of Formal Logic
Volume XIII, Number 3, July 1972
NDJFAM

AXIOMATIC, SEQUENZEN-KALKUL, AND SUBORDINATE PROOF VERSIONS OF S9

ARNOLD VANDER NAT

1.1 *The System* S9. In [8] the system S9 was presented with primitive connectives $\sim$, &, and $\strictif$ as S3 plus the axioms

(*a*) $\sim p \vee ((\sim (p \,\&\, \sim p) \strictif p) \vee (p \strictif (p \strictif p)))$,
(*b*) $\sim p \vee ((p \strictif p) \strictif p)$,
(*c*) $(p \strictif p) \strictif \sim (\sim (p \strictif p) \strictif (p \strictif p))$,

the rules being Substitution, Strict Detachment, and Adjunction.[1] A simpler formulation of S9 can be given, however, in that (*b*) is redundant and (*a*) and (*c*) can be simplified. If we abbreviate $\sim x \strictif x$ by $\Box x$ and $\sim (x \,\&\, \sim y)$ by $x \supset y$, then in S3, $x \strictif y$ is strictly equivalent (s.e.) to $\Box (x \supset y)$ and $x \strictif x$ is s.e. to $\Box \mathrm{t}$, where t is any tautology of classical two-valued logic, PC. Thus, in S3 the axioms (*a*), (*b*), and (*c*) are s.e. to

(*d*) $\sim (p \strictif \Box \mathrm{t}) \supset (p \supset \Box p)$,
(*e*) $p \supset (\Box \mathrm{t} \strictif p)$,
(*f*) $\Box \mathrm{t} \strictif \sim \Box\Box \mathrm{t}$

respectively. Now (*f*) is derivable from $\sim \Box\Box \mathrm{t}$ and (*e*), and (*e*) is derivable from $\sim \Box\Box \mathrm{t}$ and (*d*). The latter is shown as follows. The formula $(\sim p \strictif q) \,\&\, (p \strictif q) \strictif \Box q$ is provable in S3, so that $\sim \Box\Box \mathrm{t} \strictif ((\sim p \strictif \Box \mathrm{t}) \supset \sim (p \strictif \Box \mathrm{t}))$ is provable in S3. Hence, by $\sim \Box\Box \mathrm{t}$ and (*d*) we have $(\sim p \strictif \Box \mathrm{t}) \supset (p \supset \Box p)$. Substituting $\Box \mathrm{t} \supset p$ for p and detaching $\sim (\Box \mathrm{t} \supset p) \strictif \Box \mathrm{t}$, we have $(\Box \mathrm{t} \supset p) \supset (\Box \mathrm{t} \strictif p)$, which yields (*e*) by a two-valued tautology. Hence, in S3 (*a*), (*b*), and (*c*) are derivable from $\sim \Box\Box \mathrm{t}$ and $\sim (p \strictif \Box \mathrm{t}) \supset (p \supset \Box p)$, and *vice versa.* A simpler formulation of S9 in $\sim$, &, and $\strictif$ is thus S3 plus

(*g*) $\sim (\sim (p \strictif p) \strictif (p \strictif p))$
(*h*) $\sim (p \strictif (p \strictif p)) \supset (p \supset (\sim p \strictif p))$.

1.2 It is desirable to present yet another formulation of S9: a *Lemmon*

1. For a detailed discussion of S9 see [8].

Received December 13, 1969