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FREE S5 ALGEBRAS

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To Leon Mirsky on
his sixtieth birthday

A closure algebra is a system $\langle A, \wedge, \vee, -, 0, 1, C\rangle$ such that $\langle A, \vee, \wedge, -, 0, 1\rangle$ is a Boolean algebra with smallest element 0 and largest element 1, and C is a unary operation satisfying the identities $x \leqslant Cx$, $CCx = x$, $C(x \vee y) = Cx \vee Cy$, and $C0 = 0$. A member x of A is called closed if $Cx = x$. It is well known that the theorems of the Lewis system S4 are those formulas which are valid in every closure algebra (when we interpret the possibility operator by C). In S5, the theorems are the formulas which are valid in every closure algebra such that the complement of each closed element is closed. Let us call such a closure algebra an S5 algebra. The free closure algebra with one generator is already so complicated that its structure is unknown. However, S5 algebras are much simpler. In this note we shall determine the free S5 algebras with finitely many generators. This result was stated without proof in [1], Theorem 19.

An S5 subalgebra of an S5 algebra A is a Boolean subalgebra of A which is closed under C. Similarly an S5 homomorphism $f: A_1 \rightarrow A_2$ of S5 algebras is a Boolean homomorphism such that $f(Cx) = Cf(x)$ for all $x \in A_1$. If A is any Boolean algebra, we let A^* be the S5 algebra obtained by introducing in A the closure operation C such that $Cx = 1$ for all $x \neq 0$ and $C0 = 0$. It is known that a formula is a theorem of S5 if and only if it is valid in every S5 algebra of the form A^*. We shall not make use of this fact. Indeed it will be a consequence of our theorem proved below.

Lemma 1 *If A is a closure algebra, then A is an* S5 *algebra, if and only if $C(x \wedge y) = Cx \wedge y$ for any $x \in A$ and any closed $y \in A$.*

Proof: If A is an S5 algebra, and y is closed, then $C(x \wedge y) \leqslant Cx \wedge Cy = Cx \wedge y$. Also $x \wedge y \leqslant C(x \wedge y)$ implies $x \leqslant -y \vee C(x \wedge y)$. Since $-y$ is closed, it follows that $Cx \leqslant -y \vee C(x \wedge y)$, and so $Cx \wedge y \leqslant C(x \wedge y)$. Thus $C(x \wedge y) = Cx \wedge y$.

Conversely suppose this identity holds and y is a closed element of A. Then $0 = C(-y \wedge y) = C(-y) \wedge y$. Hence $C(-y) \leqslant -y$ and therefore $-y$ is closed.

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