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DEGREES OF PARTIAL FUNCTIONS

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In this paper we consider a new notion of relative recursion on partial functions, which allows for an easy definition of the recursive infimum of two functions.

1 Let U be the set of partial mappings: $\omega \to \{0, 1\}$. We write $f(x) = *$ if $f(x)$ is undefined. U will be the universum for our recursion theory. Computations are introduced using Kleene brackets [1]. So we have a relation

$$\{e\}^{\alpha}(\vec{m}) \cong n$$

where e, m_i, $n \in \omega$, $\alpha \in U$. A computation $\{e\}^{\alpha}(\vec{m})$ is undefined if either it never stops or it uses $\alpha(n)$ for an n s.t. $\alpha(n) = *$.

1.1 Definition α is recursive in $\beta(\alpha \leqslant \beta)$ if for some $e \in \omega$

$$\alpha \subseteq \lambda x \cdot \{e\}^{\beta}(x).$$

It is easy to see that $\leqslant$ is a transitive relation on U. We write $\alpha \equiv \beta$ if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$. Of course $\equiv$ is an equivalence relation. The equivalence classes are called degrees. The lowest degree, 0, is the degree of the partial recursive functions.

Motivation We see $\alpha \in U$ as an object containing information (concerning its arguments). If $\alpha \subseteq \beta$ then β contains at least as much information as α does. Hence we insist to have $\alpha \leqslant \beta$ in this case. A similar argument holds if $\alpha = \lambda x\{e\}^{\beta}(x)$ for some e. These two requirements generate $\leqslant$.

As the total functions are included in U, $U/\equiv$ has cardinality $2^{\aleph_0}$, *cf.* [2]. On the other hand some equivalence classes of $\equiv$ do have cardinality $2^{\aleph_0}$ themselves. It is not difficult to find α which is not equivalent to any total function. Furthermore a straightforward spoiling construction shows that there are no minimal degrees in U. Some motivation for considering U lies in the following theorem.

1.2 Definition $1 - \mathrm{sc}(\alpha)$ is the set of *total* functions recursive in α.

1.3 Theorem *Let $V \subseteq \omega^{\omega}$ be countable and closed under recursion. Then for some $\alpha \in U$ $V = 1 - \mathrm{sc}(\alpha)$.*

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