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THE NUMERAL AXIOMS

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The introduction of the numerals as individual constants of formal number theory is generally done by appeal to a pretheoretic or intuitively given concept of the succession of numbers. A typical account might run as follows:

The terms '0', '0^1', '0^{11}', . . . we shall call numerals abbreviated by (accounts frequently and mistakenly say 'denoted by') '0', '$\bar{1}$', '$\bar{2}$', ... In general, if n is a non-negative integer, we shall let '$\bar{n}$' stand in place of (mistake: 'stand for') the corresponding numeral $\ulcorner 0^{11 \cdots 1} \urcorner$, with n strokes.

What is pedagogically prior is not necessarily epistemologically prior, but certainly one is taught the numerals before one's "intuition" of the succession of numbers is "awakened." Regardless, it is possible to introduce the numerals without appeal to some intuitively given concept of the natural number sequence. The following axioms and axiom schemata may, for convenience, be given the title of 'the theory of numeral succession,' (**NS**).

The following axiom and definition schemata provide for the usual correlation of "numbers" with numerals and a characterization of the successor function. Definition of 'numeral': '0' . . . '9' are simple numerals. If n_1 and n_2 are simple numerals and $n_1 \neq$ '0' then $\ulcorner n_1\, n_2 \urcorner$ is a compound numeral. If n_1 and n_2 are compound numerals, $\ulcorner n_1\, n_2 \urcorner$ is a compound numeral. (In the following n_1 and $n_3 \neq$ '0'.)

Axiom schema (1): $\ulcorner \mathrm{z}(n_1, n_2) = n_1 n_2 \urcorner$.

Let n_1 = '12', n_2 = '3'. Then $\ulcorner \mathrm{z}(n_1, n_2) = n_1 n_2 \urcorner$ = 'z' '(' '12' ',' '3' ')' '=' '12' '3'. The use of quasi-quotes has the effect of quoting the constant contextual background for n_1 and n_2. 'z' is intended as a function constant on a par with '(', '=', and ')' as far as quasi-quotes are concerned, i.e., part of the constant context for n_1 and n_2. Thus 'z(12, 3) = 123' is an instance of axiom schema (1), and **NS** asserts that z(12, 3) = 123.

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