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SOME POST-COMPLETE EXTENSIONS OF S2 AND S3

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We shall take M, $\vee$, and $\neg$ as primitive connectives. Let $\mathcal{L}$ be the set of all wffs with these connectives. If $\alpha, \beta \in \mathcal{L}$, we shall write $\alpha \dashv \beta$ for $\neg M \neg(\neg\alpha \vee \beta)$, and $\alpha \equiv \beta$ for $\neg[\neg(\alpha \dashv \beta) \vee \neg(\beta \dashv \alpha)]$. We let **f** and **t** denote the wffs $p \wedge \neg p$ and $\neg p \vee p$, respectively. If $\alpha \in \mathcal{L}$, we denote by $\mathcal{L}[\alpha]$ the smallest subset of $\mathcal{L}$ containing α and closed under the connectives M, $\vee$, and $\neg$. A *modal logic* L is a proper subset of $\mathcal{L}$ which is closed under the rules of uniform substitution and *modus ponens*, and contains all tautologies. If L_1 and L_2 are modal logics, then L_1 is an *extension* of L_2 iff $\mathsf{L}_2 \subseteq \mathsf{L}_1$. A modal logic is called *Post-complete* if it has no proper extensions. Let $p(\mathsf{L})$ be the number of Post-complete extensions of a modal logic L. Several papers have considered the problem of evaluating $p(\mathsf{L})$, for various modal logics L [1, 2, 3]. It has long been known that $p(\mathrm{S2}) \geqslant \aleph_0$. Segerberg claims in [3] to prove that $p(\mathrm{S3}) = 2^{\aleph_0}$: his proof is incorrect, but it may easily be modified to show that $p(\mathrm{S2}) = 2^{\aleph_0}$ and that $p(\mathrm{S3}) \geqslant \aleph_0$. Whether or not $p(\mathrm{S3}) = \aleph_0$ remains an open question, to which this author believes the answer is probably affirmative. Most of the work on Post-complete systems uses the classical results of Lindenbaum and Tarski [4], and is therefore highly non-constructive. In fact, the only explicitly described Post-complete extensions of S3 in the literature known to the author are the systems S9 of [5] and F and Tr of [3]. This paper applies a variant of a theorem of Belnap and McCall [6] to construct some Post-complete extensions of the Lewis systems S2 and S3.

Let $\mathfrak{M} = \langle B, D, *\rangle$ be any matrix for a modal logic, where B is a Boolean algebra, D a set of distinguished elements, and $*$ interprets the possibility operator. Each element $\alpha \in \mathcal{L}[\mathbf{f}]$ determines an element $V_{\mathfrak{M}}(\alpha)$ of B, when interpreted in $\mathfrak{M}$ in the usual way.

Definition *The matrix $\mathfrak{M}$ is a functionally complete matrix* (**FCM**) *if:*

(i) for any $x \in B$, there is an $\alpha \in \mathcal{L}[\mathbf{f}]$ such that $V_{\mathfrak{M}}(\alpha) = x$.

and

(ii) for every $x \in B$, either $x \in D$ or $-x \in D$.

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