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THE WEAK TOPOLOGY ON LOGICAL CALCULI

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1 *Introduction* The basis of this note is the thought that the discrete topology on $\mathcal{Q} = \{0, 1\}$, the topology generally used for mathematical statements about logical calculi, "throws away too much" of what is available by retaining an ordering on two elements. Perhaps it is still possible to say something about propositional calculus, and even predicate calculus, by regarding $\mathcal{Q}$ as an ordered set.

The first section below deals with a topology induced on the propositional calculus of a set of variables which arises by making each of the usual realizations in $\mathcal{Q}$, topologized, continuous. According to Theorem 1, this is the smallest topology for which consequence-closed sets are always closed. Theorem 2 pertains to the theory induced by a set of propositional formulas, the Lindenbaum algebra of the calculus, quotient topologies and an embedding of the Lindenbaum algebra in a product of $\mathcal{Q}$'s. In the second section below, dealing with first order predicate languages, a weak topology on the formulas of such a language is induced, with the object of obtaining the first order analogue of Theorem 1. In effect, for first order languages, the topology naturally associated with the "external" or semantic notion of consequence is characterized "internally," in terms of canonical realizations only. The propositional calculus (with its own weak topology) on the atomic formulas of the language is homeomorphically embedded in the larger space of all formulas, and the new topology is the smallest fulfilling a natural satisfiability condition expressed in terms of satisfiability in canonical realizations.

2 *Propositional Calculus* Let $\mathcal{Q}$ be endowed with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$. Let P be an infinite set of propositional variables and $\mathrm{Prop}(P)$ the propositional calculus on P. Let $\mathrm{Hom}\,(\mathrm{Prop}(P), \mathcal{Q})$ be the set of realizations ρ: $\mathrm{Prop}(P) \rightarrow \mathcal{Q}$. Now, let $\mathrm{Prop}(P)$ be given the weak topology, $\mathcal{W}$, induced by the realizations ρ. Observe that, for $A \in \mathrm{Prop}(P)$, $\mathrm{Cl}_{\mathcal{W}}(\{A\}) = \{B \in \mathrm{Prop}(P) \mid A \rightarrow B \text{ is a theorem}\}$. (If one had chosen $\{0\}$, rather than $\{1\}$, to be closed in $\mathcal{Q}$, $\mathrm{Cl}_{\mathcal{W}}(\{A\})$ would have been $\{B \mid B \rightarrow A \text{ is a theorem}\}$.) As usual, define $\mathrm{Con}(S) = \{A \in \mathrm{Prop}(P) \mid A \text{ is a consequence of } S\}$, $S \subset \mathrm{Prop}(P)$. The discrete

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