Notre Dame Journal of Formal Logic
Volume XVII, Number 3, July 1976
NDJFAM

MONADS FOR REGULAR AND NORMAL SPACES

ROBERT WARREN BUTTON

Given an enlargement $*(X, \mathfrak{T})$ of a topological space $(X, \mathfrak{T})$, the monad of a point $x \in X$ is defined to be $\mu(x) = \bigcap \{*F: x \in F \in \mathfrak{T}\}$. It is known that for any space $(X, \mathfrak{T})$, the family of monads $\{\mu(x): x \in X\}$ contains all the information about $\mathfrak{T}$ in the sense that for each $x \in X$, $\{F \subseteq X: \mu(x) \subseteq *F\}$ is exactly the neighborhood filter at x. However, it is possible to say something about $\mathfrak{T}$ without resorting to this method. For example, a space X is Hausdorff iff for any two points x and y in X, $\mu(x) \cap \mu(y) = \emptyset$. In this paper some further relationships between the topology on X and $\{\mu(x): x \in X\}$ will be shown, and particularly nice characterizations of regular and normal spaces will be given. These characterizations will be in terms of a natural topology on $*X$, the Q-topology. Let us briefly consider the Q-topology.

It is possible to write a formal sentence expressing the fact that $\mathfrak{T}$ is a topology on X, so in any enlargement $*(X, \mathfrak{T})$, $*\mathfrak{T}$ is closed under *finite intersections (and hence under finite intersections) and under internal unions. $*\mathfrak{T}$ also contains $\emptyset$ and $*X$, so is the base for a topology on $*X$, the Q-topology. Sets in $*\mathfrak{T}$ are said to be *open, subsets of $*X$ whose complements are in $*\mathfrak{T}$ are said to be *closed, and so on. Robinson has shown that an internal set is *open iff it is Q-open and *closed iff it is Q-closed. Also, a standard set A is open iff $*A$ is *open. We now introduce a new type of refinement relation which is particularly suited for studying Q-topologies.

Definition 1 We shall say that the covering $\mathfrak{U}_1$ of X fills the covering $\mathfrak{U}_2$ of X if for each $V \in \mathfrak{U}_2$, $V = \bigcup \{U \in \mathfrak{U}_1: U \subseteq V\}$.

Let $\mathfrak{G}$ be the collection of all finite open coverings of a given space X and let FR be the filling relation restricted to $\mathfrak{G} \times \mathfrak{G}$. The left domain of FR is $\mathfrak{G}$ since every covering fills itself and for each finite collection $\mathfrak{U}_1, \ldots, \mathfrak{U}_n$ of coverings in $\mathfrak{G}$, $\{U_1 \cap \ldots \cap U_n: U_1 \in \mathfrak{U}_1, \ldots, U_n \in \mathfrak{U}_n\}$ is a covering in $\mathfrak{G}$ filling each of $\mathfrak{U}_1, \ldots, \mathfrak{U}_n$, so the relation FR is concurrent. Hence, there is a covering of $*X$ in $*\mathfrak{G}$, say φ_F, such that if $\mathfrak{U}$ is a finite open covering of X, φ_F fills $*\mathfrak{U}$. In general φ_F is not unique and we shall speak of an arbitrary but fixed φ_F. For each $x \in *X$, $\{P \in \varphi_F: x \in P\}$ is an

Received December 13, 1974