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A NOTE ON THE ADEQUACY OF TRANSLATIONS

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Given a theory which can be axiomatized in more than one way using different sets of constants as primitive, any two such axiomatizations, *sans* definitions, can be said to be deductively (semantically) synonymous, following de Bouvêre [1], in the sense that the addition of appropriate definitions to each and closure under provability (consequence) results in a single theory. The question then arises: Can the definitions of one axiomatization be used to provide another axiomatization by translating the first into the second?

For the analagous problem concerning formal systems, Hiż [3] signalled that this is not always the case. In [2], Halmos takes the Hilbert-Ackermann axioms for a sentential logic of $\sim$ and $\vee$, and the rule of inference

$$\frac{\begin{array}{c} p \vee q \\ \sim p \end{array}}{q}$$

and provides an axiom system for $\sim$ and & by means of the definition

$$p \vee q \leftrightarrow \sim(\sim p \; \& \; \sim q).$$

Hiż gives a model-theoretic demonstration that the translated system is incomplete.

The general theorem which could have predicted this result is as follows:

If $\mathrm{T}(A)$ *is the closure of a formal system in a language* $\mathcal{L}$, *with axioms* A1, . . ., AN; *and rules* R1, . . ., RM; *and* $\mathbf{t}$ *a rule of translation from* $\mathcal{L}$ *to* $\mathcal{L}'$, *then* T', *the closure of* $\mathbf{t}$(A1), . . ., $\mathbf{t}$(AN), $\mathbf{t}$(R1), . . ., $\mathbf{t}$(RM), *is equal to* $\mathbf{t}(\mathrm{T}(\mathrm{A}))$.

To show this, note that if RJ is a k-place rule, it contains the $k+1$-tuple $\langle y_1, \ldots, y_k, x\rangle$ just when $\mathbf{t}(\mathrm{R}J)$ contains $\langle \mathbf{t}(y_1), \ldots, \mathbf{t}(y_k), \mathbf{t}(x)\rangle$. So

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