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A FIRST-ORDER LOGIC OF KNOWLEDGE AND BELIEF WITH IDENTITY. II

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Part I* presented a formal first-order, Gentzen-type system of the logic of knowledge and belief based closely upon the informal work of Hintikka [1]. In Part II, this system is shown to be semantically complete.

8 *Completeness: Preliminary Definitions* Formula-sequences can be regarded as finite sequences of formulae, and sequents as ordered pairs of such sequences. In what follows we shall accordingly deal with sequences rather than with expressions, though we shall retain the expression symbolism already introduced.

If σ is a finite sequence of length n (an n-tuple) and $0 \leqslant k < n$, then $(\sigma)_k$ shall be the $k + 1$'st element of σ; if α and β are such that $((\sigma)_\alpha)_\beta$ is defined, we write $(\sigma)_{\alpha*\beta} = ((\sigma)_\alpha)_\beta$; as before, $\alpha*\beta*\gamma = (\alpha*\beta)*\gamma$. Let $\mathrm{D}(\Gamma, i) \equiv \forall j[([j < i] \supset [((\Gamma)_{j*0} = \mathbf{K}) \vee ((\Gamma)_{j*0} = \mathbf{B})]) \;\&\; ([i \leqslant j] \supset [((\Gamma)_{j*0} \neq \mathbf{K}) \;\&\; ((\Gamma)_{j*0} \neq \mathbf{B})])]$. If $X = \{\Gamma | \exists i \mathrm{D}(\Gamma, i)\}$, define $\delta : X \to \omega$ by $\delta(\Gamma) = i$ iff $\mathrm{D}(\Gamma, i)$. Let $Y = \{S | -\exists i \exists j[(S)_{0*i} = (S)_{1*j}] \;\&\; -\exists i[((S)_{1*i*0} = \mathsf{I}) \;\&\; ((S)_{1*i*1} = (S)_{1*i*2})]\}$ and $Z = \{S \epsilon Y | ((S)_0 \epsilon X) \;\&\; ((S)_1 \epsilon X)\}$. Finally, if Γ has length $n + 1$ and $0 \leqslant j \leqslant n$, then $\Phi(\Gamma, i)$ is the sequence of formulae defined by $\forall k[([k < i] \supset [(\Gamma)_k = (\Phi(\Gamma, i))_k]) \;\&\; ([i \leqslant k] \supset [(\Gamma)_{k+1} = (\Phi(\Gamma, i))_k])]$.

We now define a number of functions f from sequents to sequents such that $f(S)$ is related to S as premiss to conclusion by a rule of inference, modulo applications of the enabling rules. These functions will be used to construct the 'proof trees' used in the completeness argument. We assume throughout this section that a_k is the k'th free individual variable.

1. f_0 shall be the identity function on the set of sequents.
2. If $Z_{\mathbf{N},0} = \{S \epsilon Z | \exists i((S)_{0*i*0} = \mathbf{N})\}$, then $f_{\mathbf{N},0}$ is defined on $Z_{\mathbf{N},0}$ by $f_{\mathbf{N},0}(S) = \langle \Phi((S)_0, i), \Gamma \rangle$, where

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