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AN INDEPENDENT STATEMENT ABOUT METRIC SPACES

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In a metric space can the points near some point x pack close to each other with ever-increasing density (in the sense of cardinality or power) as x is approached, or must it always be the case that this density reaches a maximum at a certain distance from x and does not increase for smaller distances? We give a precise definition of this density concept and show that the former case can happen (for a space of cardinality $\aleph_\omega$) but that the question as to whether it can happen in a space of power less than or equal to that of the continuum cannot be answered. Our results are based on some of the recent independence results in set theory.

1 *Preliminaries* In the following (X, ρ) will be a metric space, A a subset of X, and x a point of X. We define the A-packing power near x by

$\mathrm{P}_A(x) = \sup \{a \leqslant \mathrm{card}\, X \mid \exists \varepsilon > 0, \mathrm{card}[(\mathrm{S}(x, \varepsilon_2) - \mathrm{S}(x, \varepsilon_1)) \cap A] \geqslant a$ for all ε_1, ε_2 satisfying $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon\}$.

That is, if C denotes the set of cardinals $a \leqslant \mathrm{card}\, X$ such that between any two small enough concentric spheres about x there lie at least a points of A, then $\mathrm{P}_A(x) = \sup C$. $\mathrm{P}_X(x)$ will be written $\mathrm{P}(x)$ and called the packing power near x. A packed point of X is a point x for which $\mathrm{P}(x) > 0$. A packed space is one whose points are all packed. It is easily seen that a packed space is perfect, but that a perfect space need not be packed (take an appropriate subspace of $\mathcal{R}^1$). We remark that $\mathrm{P}_A(x)$ measures how close (in the sense of cardinality) to *each other* the points near x are packed, not how closely they pack about x itself. We will discuss this other question later.

The question here is whether or not it is always the case (i.e., for all X, A, x) that $\mathrm{P}_A(x) \in C$, i.e., whether $\sup C \in C$. We will work in Zermelo-Fraenkel set theory (**ZF**) including the axiom of choice. We will also make use of the results on the status of the continuum hypothesis (**CH**) and the generalized continuum hypothesis (**GCH**) in **ZF**, established by K. Gödel [1] and by P. J. Cohen [2]. In particular we note that $2^{\aleph_0} = \aleph_{\omega+1}$ is consistent with **ZF** [3]. The assertion $\overline{\sup C \in C}$ is taken to mean: For all (X, ρ) and for

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