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On Uncountable Boolean Algebras With No Uncountable Pairwise Comparable or Incomparable Sets of Elements

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Elements a, b, of a Boolean algebra are said to be *comparable* iff either $a \leqslant b$ or $b \leqslant a$, otherwise *incomparable*. A *chain* in a Boolean algebra is a set of pairwise comparable elements, while a *pie* is a set of pairwise incomparable elements.

In [2] Baumgartner and Komjath proved, using $\diamondsuit_{\aleph_1}$:

Theorem 1 (Baumgartner-Komjath) *Assume* $\diamondsuit_{\aleph_1}$. *There is an uncountable Boolean algebra with no uncountable chain or pie.*

In [6] Rubin, also using $\diamondsuit_{\aleph_1}$, proved:

Theorem 2 (Rubin) *Assume* $\diamondsuit_{\aleph_1}$. *There is a Boolean algebra* B, *with* $\overline{\overline{B}} = \aleph_1$, *in which every ideal is* $\aleph_0$*-generated and every subalgebra is generated by an ideal and* $\aleph_0$ *elements. Thus,* B *has only* $\aleph_1$ *ideals and subalgebras.*

Using only *CH*, Berney and Nyckos [3] and Bonnet [4] proved:

Theorem 3 *Assume CH. There is an uncountable Boolean algebra with no uncountable pie.*

They chose a set A of reals of cardinality $\aleph_1$, and the Boolean algebra is the Boolean algebra of subsets of the reals generated by (r, s), $r, s \in A$.

In the opposite direction, Baumgartner [1] showed:

Theorem 4 *It is consistent with ZFC that* $2^{\aleph_0} = \aleph_2$, *Martin's axiom holds, and every Boolean algebra of cardinality* $\aleph_1$ *contains an uncountable pie.*

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