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Consequences, Consistency, and Independence in Boolean Algebras

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Introduction In this paper we work within an arbitrary but fixed Boolean algebra $(B, +, \cdot, ', 0, 1)$ and with vectors $x = (x_1, \ldots, x_n) \in B^n$, $y = (y_1, \ldots, y_m) \in B^m$, where n and m are two arbitrary but fixed positive integers.* A Boolean function $f: B^n \to B$ is characterized by the Boole expansion theorem [1], [2][1]

$$(1) \qquad f(x_1, \ldots, x_n) = \sum_{(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n} f(\alpha_1, \ldots, \alpha_n) x_1^{\alpha_1} \ldots x_n^{\alpha_n} ,$$

where $\sum$ denotes iterated sum (disjunction) when the vector $(\alpha_1, \ldots, \alpha_n)$ runs over $\{0, 1\}^n$ and $x^0 = x'$, $x^1 = x$. If each $(\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$ is interpreted as a number $i \in \{0, \ldots, 2^n - 1\}$ written in basis 2 and the corresponding minterm $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ is denoted by $m_i(x)$, formula (1) becomes

$$(2) \qquad f(x) = \sum_{i=0}^{2^n-1} f(i) m_i(x) \; .$$

In particular a Boolean function $r: B^{n+m} \to B$ admits the expansions

$$(3) \qquad \begin{aligned} r(x, y) &= \sum_{i=0}^{2^n-1} r(i, y) m_i(x) \\ &= \sum_{j=0}^{2^m-1} r(x, j) m_j(y) \\ &= \sum_{i=0}^{2^n-1} \sum_{j=0}^{2^m-1} r(i, j) m_i(x) m_j(y) \end{aligned}$$

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