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Equivalence Relations and S5

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1 An equivalence relation is commonly defined as one which is reflexive, symmetrical, and transitive. This paper* starts from the problem of finding a pair of conditions on a dyadic relation which together yield equivalence but neither of which by itself yields either reflexiveness or symmetry or transitivity. It will be shown that there are infinitely many such pairs of conditions.

There is a parallel problem in modal logic, that of finding a pair of formulas which, if added to the minimal normal modal logic K, yield precisely $S5$, but neither of which, when added to K, yields either $Lp \supset p$ or $p \supset LMp$ or $Lp \supset LLp$ as a theorem. It will be shown that there are infinitely many such pairs of formulas.

2 One solution to the second problem is provided by the following formulas:

$A \quad LMLp \supset p$
$B \quad MLp \supset LMLLp.$

Since in $S5$ an affirmative modality is equivalent to its last member, it is clear that A and B are theorems of $S5$ and hence that $S5$ contains $K + A + B$. For the converse it is sufficient to derive $MLp \supset Lp$ and $Lp \supset p$. We first note that A is interdeducible in the field of K with its dual:

$A' \quad p \supset MLMp.$

We then have:

$MLp \supset Lp$ $\quad [B, A(Lp/p) \times \text{Syll}]$
$Lp \supset p$ $\quad [A'(Lp/p), B(MLp/p), A(LMLp/p), A \times \text{Syll}]$

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