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CONSTRUCTIVELY NONPARTIAL RECURSIVE FUNCTIONS

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Rose and Ullian [3] called a total function $f(x)$ constructively nonrecursive iff for some recursive function $g(x)$, $f(g(n)) \neq \varphi_n(g(n))$ for all $n \in N$, where $\varphi_n(x)$ is the partial recursive function with index n. We define a partial function $f(x)$ to be constructively nonpartial recursive iff for some recursive $g(x)$, $f(g(n)) \not\simeq \varphi_n(g(n))$, where $\simeq$ is equality for partial functions. We say that $f(x)$ is constructively nonpartial recursive via $g(x)$. Note that for total functions, the two concepts coincide.

An example of a constructively nonpartial recursive function which is a total function is:

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x) \text{ is defined} \\ 0 & \text{otherwise} \end{cases} .$$

Indeed, letting $g(x) = x$, we have

$$f(g(n)) = f(n) = \begin{cases} \varphi_n(n) + 1 \not\simeq \varphi_n(n) = \varphi_n(g(n)) & \text{if } \varphi_n(n) \text{ is defined} \\ 0 \not\simeq \varphi_n(n) = \varphi_n(g(n)) & \text{otherwise} \end{cases}$$

As an example of a constructively nonpartial recursive function which is not total, we have:

$$h(x) = \begin{cases} \text{undefined} & \text{if } \varphi_x(x) \text{ is defined} \\ x & \text{otherwise} \end{cases}$$

$h(x)$ is constructively nonpartial recursive via $g(x) = x$.

The theory of constructively nonpartial recursive functions is intimately connected with the theory of productive sets. As an analogue to the fact that any 1-1 recursive function is the productive function for some set, we have the following:

Theorem 1 *For every* 1-1 *recursive function* $g(x)$, *there is a function* $f(x)$ *which is constructively nonpartial recursive via* $g(x)$.

Proof: Suppose $g(x)$ is a 1-1 recursive function. Let $g^{-1}(x) = (\mu y)(g(y) = x)$; $g^{-1}(x)$ is partial recursive. Define

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