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SEMI-MONOTONE SERIES OF ORDINALS

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The task of calculating the sum of a series of ordinals is greatly facilitated by the absorption law, and with regard to the use of this law, increasing series of ordinals provide the example par excellence: if you can only keep going long enough, you can forget about some of the early terms. In this note* we define a generalization of the increasing series, and by means of this generalization we obtain some conditions under which a series of ordinals has minimal sum.

Let $s = (s_\xi)_{\xi<\alpha}$ be a sequence of ordinals: we denote by "$o(s)$" the ordinal α—the "order-type" of the sequence—and by "$\Sigma(s)$" the sum of the associated series. A second sequence $t = (t_\zeta)_{\zeta<o(t)}$ is called a "permutation" of s if $o(t) = o(s)$ and there is a bijection $f: o(s) \to o(t)$ such that $s_\xi = t_{f(\xi)}$ for all $\xi < o(s)$. In some of our previous papers ([1], [2], [3]) we have been interested in the set $S(s) = \{\Sigma(t);\ t \text{ is a permutation of } s\}$, and in [3] we had occasion to introduce the concept of a semi-monotone sequence. A sequence $s = (s_\xi)_{\xi<\alpha}$ of ordinals is called "semi-monotone" if for each pair (μ,ν) of ordinals μ, $\nu < \alpha$ there is a third ordinal $\psi < \alpha$ such that $\nu \leqslant \psi$ and $s_\mu \leqslant s_\psi$. We showed in [3] that if s is a semi-monotone sequence of positive ordinals with $o(s)$ a regular initial ordinal, then for any permutation t of s we have t semi-monotone and $\Sigma(t) = \Sigma(s)$. Since we shall be mainly concerned with series of ordinals, we make the blanket assumption that unless the contrary is either obvious or explicitly stated, all sequences referred to henceforth will have only positive terms.

Let r, s be any two sequences. We define $r \dot{\cup} s$ to be the sequence t such that $o(t) = o(r) + o(s)$, $t_\xi = r_\xi$ for $\xi < 0(r)$, and $t_{o(r)+\zeta} = s_\zeta$ for $\zeta < o(s)$. If $s = r \dot{\cup} t \dot{\cup} u$ for some sequences r, t, u, then we call t a "segment" of s, and say that t is initial (final) if $o(r) = 0(o(u) = 0)$. For any sequence s with $o(s) \neq 0$, we put $\sup(s) = \sup\{s_\xi;\ \xi < 0(s)\}$. Finally, we say that s is non-empty if $o(s) \neq 0$.

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