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Some Remarks on Equivalence in Infinitary and Stationary Logic

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The logic $L(aa)$ is obtained by adding a quantifier aa ("almost all") ranging over countable sets (see [1]). If instead one extends first-order finitary logic by allowing arbitrary conjunctions and disjunctions and quantification $\exists\langle x_\alpha : \alpha < \lambda\rangle$ and $\forall\langle x_\alpha : \alpha < \lambda\rangle(\lambda < \kappa)$ and restricts to formulas with fewer than κ free variables, one obtains the infinitary logic $L_{\infty\kappa}$ (see, for example, [3]). The following theorem extends Section 5 of [1] and answers a question of Nadel.

Theorem 1 *For every cardinal κ, there are structures A and B such that A and B satisfy the same sentences of $L_{\infty\kappa}$ but not of $L(aa)$.*

Proof: Fix $\kappa > \omega$. It is routine to construct a chain $\langle A_\alpha : \alpha < \kappa^+\rangle$ of structures for the vocabulary $\{X, Y, P, \epsilon\}$, satisfying the following inductive hypotheses (1) through (4) below. Here X, Y, and P are unary relation symbols and ϵ is binary. We write $A_\alpha = (A_\alpha; X_\alpha, Y_\alpha, P_\alpha, \in)$, and we use the standard notation $[Z]^\omega$ for the set of countably infinite subsets of a set Z.

(1) $A_\alpha = X_\alpha \cup Y_\alpha$, $Y_\alpha = [X_\alpha]^\omega$, $P_\alpha \subseteq Y_\alpha$, and $\in$ is the membership relation on $X_\alpha \times Y_\alpha$. Also $\alpha < \beta$ implies $A_\alpha \subsetneq A_\beta$.

(2) $|X_\alpha| = 2^\kappa$.

(3) Suppose that $Z_0 \subseteq X_\alpha$ and $|Z_0| \leq \kappa$, and that $|Z| = \kappa$ and $(Z_0 \cup [Z_0]^\omega, P_\alpha \cap [Z_0]^\omega, \in) \subseteq (Z \cup [Z]^\omega, P, \in)$. Then for some $Z' \subseteq X_{\alpha+1}$, there is an isomorphism j from $(Z \cup [Z]^\omega, P, \in)$ onto $(Z' \cup [Z']^\omega, P_{\alpha+1} \cap [Z']^\omega, \in)$ such that j extends the identity function on $Z_0 \cup [Z_0]^\omega$.

(4) If α is a limit ordinal of cofinality ω, then for all $s \in \left[\bigcup_{\beta<\alpha} X_\beta\right]^\omega - \bigcup_{\beta<\alpha} [X_\beta]^\omega$, we have $s \in P_\alpha$.

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