Notre Dame Journal of Formal Logic
Volume 25, Number 4, October 1984

Extensionality in Bernays Set Theory

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Gandy has shown, in [3], that the consistency of Bernays-Gödel set theory can be reduced to that of the theory without the axiom of extensionality. We verify a parallel result for the theory presented in [1], Appendix. Since one of the axioms (schemes) in [1] is the reflection principle, which is of rather different character than other axioms about set existence, some new considerations are required. Also we take a "top down" approach, which seems to work more quickly in this context than Gandy's "bottom up" one. At the end, we comment on other set theories with regard to extensionality and the "top down" approach.

The theory **B** is a single-sorted first-order theory with a binary predicate $\in$, a monadic function symbol σ, and a term forming operator $\{x/\ldots\}$. Using obvious abbreviations like $a \subseteq b$ and $a \cap b$, the nonlogical axioms of **B** can be stated as follows:

(Ex) [The axiom of extensionality] $a \simeq b \,\&\, a \in c \rightarrow b \in c$, *where* '$a \simeq b$' *is for* '$a \subseteq b \,\&\, b \subseteq a$'.
(CF) [The axioms of choice and Fundierung] $a \in c \rightarrow [\sigma(c) \in c \,\&\, a \notin \sigma(c)]$.
(Cp) [The axiom of impredicative comprehension] $c \in \{x/\phi(x)\} \leftrightarrow S(c) \,\&\, \phi(c)$, *where* '$S(c)$' *is for* '$\exists z c \in z$'.
(Rf) [The axiom of reflection] $\phi \rightarrow \exists y[ST(y) \,\&\, S(y) \,\&\, \phi^y]$.

Here, '$ST(y)$' is for '$\forall u, v(v \in y \,\&\, (u \subseteq v \vee u \in v) \rightarrow u \in y)$ [y is strongly transitive], and ϕ^y is the result of relativization of ϕ to y, i.e., any free variable a in ϕ is replaced by $a \cap y$ unless $S(a)$ is given, $\forall x \psi(x)$ is replaced by $\forall x(x \subseteq y \rightarrow \psi^y(x))$, $\exists x \psi(x)$ by $\exists x(x \subseteq y \,\&\, \psi^y(x))$, and $\{x/\psi(x)\}$ by $\{x/x \in y \,\&\, \psi^y(x)\}$.

(Eq) [*An axiom of equality*] $a \simeq b \rightarrow \sigma(a) \simeq \sigma(b)$.
(Em) $\neg \exists x\, x \in a \rightarrow \neg \exists x\, x \in \sigma(a)$.

(Actually, the last two axioms do not appear in [1]. Indeed, (Eq) is provable from (Ex), (Cp), and (Rf). The last determines the value of σ at $a = \varnothing$, which

Received September 26, 1983; revised January 23, 1984