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An Axiomatization of the Equivalential Fragment of the Three-Valued Logic of Łukasiewicz

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The problem of axiomatizing the purely equivalential fragment of the infinite-valued Łukasiewicz logic (L_∞) and the corresponding variety of algebras remains open. Moreover for every $n = 3, 4, \ldots$ one may ask about axiomatization of the purely equivalential fragment of n-valued Łukasiewicz logic (L_n). In this paper we give an axiomatization of the purely equivalential fragment of L_3 and an appropriate set of identities determining the corresponding variety of algebras (see [3]).

Let us recall that the three-valued logic of Łukasiewicz L_3 is determined by the following matrix: $L_3 = (\{0,1,2\}, \{0\}, \rightarrow_L, \wedge_L, \vee_L, \sim_L)$ where $x \rightarrow_L y = max(0, y-x)$, $x \wedge_L y = max(x,y)$, $x \vee_L y = min(x,y)$, and $\sim_L x = x \rightarrow_L 2$ (see [5]).

The other well-known three-valued logic is the logic H_3 considered by Heyting in [1]. It is determined by the matrix $H_3 = (\{0,1,2\}, \{0\}, \rightarrow_H, \wedge_H, \vee_H, \sim_H)$, where $x \rightarrow_H y = y$ whenever $x < y$ and $x \rightarrow_H y = 0$ otherwise, $x \wedge_H y = max(x,y)$, $x \vee_H y = min(x,y)$, and $\sim_H x = x \rightarrow_H 2$.

Let the symbols $L_3^{\equiv}$ and $H_3^{\equiv}$ denote the purely equivalential fragments in question. Since $x \equiv y =_{df} (x \rightarrow y) \wedge (y \rightarrow x)$ then $L_3^{\equiv}$ and $H_3^{\equiv}$ are determined by the following matrices $\mathbf{L}_3^{\equiv}$ and $\mathbf{H}_3^{\equiv}$ respectively: $\mathbf{L}_3^{\equiv} = (\{0,1,2\}, \{0\}, \equiv_L)$ where $x \equiv_L y = max(x-y, y-x)$ and $\mathbf{H}_3^{\equiv} = (\{0,1,2\}, \{0\}, \equiv_H)$ where $x \equiv_H y = max(x,y)$ whenever $x \neq y$ and $x \equiv_H y = 0$ otherwise.

It is known that neither $L_3 \not\subseteq H_3$ nor $H_3 \not\subseteq L_3$; for example $(\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta) \in H_3 - L_3$ whereas $((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha) \in L_3 - H_3$. Nevertheless we shall prove that the purely equivalential fragments of L_2 and H_3 are identical.

The equality $L_3^{\equiv} = H_3^{\equiv}$ is an immediate consequence of the fact that the matrices $\mathbf{L}_3^{\equiv}$ and $\mathbf{H}_3^{\equiv}$ are isomorphic. The reader will have no difficulty in verifying that the required isomorphism is the mapping $i: \{0,1,2\} \rightarrow \{0,1,2\}$, such that $i(0) = 0$, $i(1) = 2$, $i(2) = 1$.

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