Notre Dame Journal of Formal Logic
Volume 25, Number 3, July 1984

A Model Theoretic Proof of Feferman's Preservation Theorem

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Let L be a countable first-order language containing a binary relation symbol $\triangleleft$. If $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures and $\mathfrak{A} \subseteq \mathfrak{B}$, then we say $\mathfrak{B}$ is a *faithful extension* of $\mathfrak{A}$ if and only if for any $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ if $\mathfrak{B} \models b \triangleleft a$, then $b \in \mathfrak{A}$. Thus if $\triangleleft$ is a linear order on $\mathfrak{A}$, $\mathfrak{B}$ is a faithful extension if and only if it is an end extension.

In [2] Feferman gives a very natural classification of the formulas which are preserved under faithful extensions. His proof uses a many-sorted interpolation theorem proved by a cut elimination argument. With the introduction of recursively saturated models Barwise and Schlipf [1], and Schlipf [5] attempted to give a unified framework for many preservation and definability theorems. In this note I will give an instructive model theoretic proof of Feferman's theorem. (I should note that Stern [8] and Guichard [4] have given model theoretic proofs of Feferman's theorem using model-theoretic forcing and consistency properties, respectively, but neither of these approaches matches the elegance of [5].)

The proof given here is directly inspired by Friedman's theorem [3] that every countable model of Peano Arithmetic is isomorphic to a proper initial segment of itself and the related embedding results presented in Smoryński [6]. In fact, independently of the author, Smorynski [7] uses Friedman's theorem to prove Feferman's result in the special case that $\mathfrak{A}$ and $\mathfrak{B}$ are models of Peano Arithmetic.

1 Embedding recursively saturated models

Definition 1.1 Let L be as above. We inductively define Σ a class of L-formulas as follows:

(i) If $\varphi(\bar{v})$ is quantifier free, then $\varphi(\bar{v})$ is in Σ.
(ii) If $\varphi(\bar{v})$ and $\psi(\bar{v})$ are in Σ, then $\varphi(\bar{v}) \wedge \psi(\bar{v})$ and $\varphi(\bar{v}) \vee \psi(\bar{v})$ are in Σ.

Received August 8, 1983; revised December 2, 1983