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The Connective of Necessity of Modal Logic S_5 is Metalogical

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Let a,b be formulas of the language of the classical propositional calculus and let the first of them be a classical thesis while the second is not. This fact is often denoted as follows: $\vdash a$, $\dashv b$. In a certain sense the operations $\vdash$ and $\dashv$ are inconsistent and we will write informally $\vdash = \neg \dashv$ ($\neg$ being negation). We can consider the operation $\vdash$ as a connective A of some propositional calculus containing the classical one and containing formulas Aa and $\neg Ab$ as theses. Among its theses would be the formulas $Aa \equiv p \rightarrow p$ and $Ab \equiv \neg(p \rightarrow p)$ (p being a propositional variable). It seems that by such a definition (i.e., $Aa = p \rightarrow p$ iff a is a thesis and $Aa = \neg(p \rightarrow p)$ iff a is not a thesis) this new logic could be obtained. This is not so, however, for among the expressions $\neg Ap$, $\neg A(p \rightarrow p)$ the first would be a thesis and the second a nonthesis, which would not allow us to treat p as a variable. We are thus led to consider the greatest such set of formulas closed under substitution, i.e., the set S defined below. This is an intuitive way to summarize the problem of this paper, i.e., the problem of building a system using the connective of assertion A and containing the classical logic.

This system will be shown to be identical with the system of modal logic S_5. The manner of introducing the connective A suggests it possesses a metalogical character in comparison with the classical connectives. This allows us to suppose that in S_5 it will be possible to "express" certain metalogical properties of the logic obtained by omitting the connective A, i.e., classical logic. Indeed, Pogorzelski's Theorem on structural completeness of classical propositional calculus (Theorem 2) is "expressed" as a rule of S_5. We also note the fact that Stone's Theorem is in the same sense equivalent to a rule of S_5^* (see below). Eventually we give some fragmentary methods of rejection of formulas in S_5 based only on classical logic and on the manner of reading the connective A.[1]

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