Notre Dame Journal of Formal Logic
Volume 24, Number 1, January 1983

# A Note on Conway Multiplication of Ordinals

JOHN HICKMAN*

We denote by '$\omega$' the first transfinite ordinal, and by '$\alpha + \beta$', '$\alpha\beta$', and '$\alpha^\beta$', respectively, the usual ordinal sum, product, and exponentiation of an ordinal $\alpha$ by an ordinal $\beta$. We assume that the reader is familiar with Cantor's $\omega$-normal form theorem, which uniquely represents a nonzero ordinal as a sum of powers of $\omega$. Given a nonzero ordinal $\alpha$, we let $\ell(\alpha)$ be the number of summands in the $\omega$-normal form of $\alpha$, and express this form as $\Sigma\{\omega^{e(\alpha,i)}c(\alpha,i); i < \ell(\alpha)\}$.

Regrettably, there seems to be no "nice" way of formulating an adequate definition of "natural" ordinal addition, and so we shall have to use the following not-so-nice way.

Let $\alpha, \beta$ be ordinals. If $\alpha\beta = 0$, set $\alpha \mathbin{\mathring{+}} \beta = \alpha + \beta$; otherwise set $\alpha \mathbin{\mathring{+}} \beta$ equal to the unique ordinal $\gamma$ whose $\omega$-normal form has the following properties:

(1) $\{e(\gamma,i); i < \ell(\gamma)\} = \{e(\alpha,j); j < \ell(\alpha)\} \cup \{e(\beta,k); k < \ell(\beta)\}$.

(2) (a) If $e(\gamma,i) = e(\alpha,j)$ for some $j < \ell(\alpha)$ but $e(\gamma,i) = e(\beta,k)$ for no $k < \ell(\beta)$, then $c(\gamma,i) = c(\alpha,j)$.

(b) If $e(\gamma,i) = e(\beta,k)$ for some $k < \ell(\beta)$ but $e(\gamma,i) = e(\alpha,j)$ for no $j < \ell(\alpha)$, then $c(\gamma,i) = c(\beta,k)$.

(c) If $e(\gamma,i) = e(\alpha,j) = e(\beta,k)$ for some $j < \ell(\alpha)$ and $k < \ell(\beta)$, then $c(\gamma,i) = c(\alpha,j) + c(\beta,k)$.

We can now define Conway multiplication, denoted by '$\times$', which was introduced by Gonshor in [1] and attributed by him to Conway.

If $\alpha\beta = 0$, then we set $\alpha \times \beta = 0$; otherwise we set $\alpha \times \beta = (\alpha \times \delta) \mathbin{\mathring{+}} \alpha$ if $\beta = \delta + 1$ for some $\delta$, and $\alpha \times \beta = \sup\{\alpha \times \delta; \delta < \beta\}$ if $\beta$ is a limit ordinal.

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*The work contained in this paper was done while the author was a research officer at the Australian National University.

*Received August 3, 1981*