Notre Dame Journal of Formal Logic
Volume 24, Number 1, January 1983

Generalized Archimedean Fields

JOHN COWLES and ROBERT LaGRANGE

We consider (linearly) ordered fields, F (actually $\langle F, +, x, 0, 1, \leqslant\rangle$). A subset $S \subseteq F$ is *positive* if $x > 0$ for $x \in S$. S is *separated* (of size a) if $a > 0$ and $|x - y| > a$ whenever $x, y \in X$. An easy exercise shows that F is Archimedean if and only if: (1) there is an infinite positive separated subset of F and (2) every bounded positive separated subset is finite.

Let κ be an infinite cardinal number. We define F to be *κ-Archimedean* in case: (1) F has a positive separated subset of cardinality κ, and (2) no positive separated subset of F having cardinality κ is bounded. An ordered field is Archimedean if and only if it is $\aleph_0$-Archimedean.

In [6] Sikorski gives a very natural example of a κ-Archimedean field $\mathcal{L}_\kappa$. To construct $\mathcal{L}_\kappa$ and related systems we use ordinal numbers with Hessenberg natural operation $\#$ (addition) and $\divideontimes$ (multiplication). Recall that an ordinal α can be uniquely written in base ω, as $\alpha = \omega^{\alpha_1}\cdot n_1 + \ldots \omega^{\alpha_s}\cdot n_S$ where $\alpha_1 > \alpha_2 > \ldots \alpha_S$ are ordinal numbers, S is finite, $n_1, \ldots, n_S$ are positive integers, "+" and "$\cdot$" are ordinal addition and multiplication respectively. This representation is called Cantor normal form. The natural sum of two ordinals in Cantor form is obtained by adding them as if they were polynomials in ω. The natural product, likewise, is obtained by multiplying them as polynomials in ω, but with the provision that natural sum is used for addition of exponents. "$\#$" and "$\divideontimes$" are commutative and associative, have identities "0" and "1" respectively. "$\divideontimes$" distributes over "$\#$".

Considering κ to be the set of ordinals less than κ, we define $\mathcal{N}_\kappa = \langle\kappa, \#, \divideontimes, \leqslant\rangle$, $\mathcal{J}_\kappa$ is the ring of differences over $\mathcal{N}_\kappa$, $\mathcal{L}_\kappa$ is the quotient field of $\mathcal{J}_\kappa$, and $\mathcal{R}_\kappa$ is the real closure of $\mathcal{L}_\kappa$. In [6] Sikorski proves that $\mathcal{L}_\kappa$ is a κ-Archimedean field of cardinality κ. Also every ordered field of cofinality κ contains a subfield isomorphic to $\mathcal{L}_\kappa$.

In case κ is regular we note that any κ-Archimedean field does have cofinality κ, also that $\mathcal{R}_\kappa$ is κ-Archimedean. An Archimedean field of cardinality λ exists if and only if $\aleph_0 \leqslant \lambda \leqslant 2^{\aleph_0}$. For uncountable κ we are interested

Received October 20, 1981; revised June 28, 1982