Notre Dame Journal of Formal Logic
Volume 24, Number 1, January 1983

Sequential Compactness and the Axiom of Choice

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A theorem is effective iff it is proved in ZF^o, where ZF^o is Zermelo-Fraenkel set theory without the axioms of choice and foundation (regularity). A well known effective theorem of F. Riesz states that a Hilbert space is finite dimensional iff its closed unit ball is compact. This may fail for sequential compactness. If U is a set of urelements, equipped with the structure of l_2, I is the ideal of all finite subsets of U and G is the group of unitary operators, by an argument similar to [7]. In the resulting permutation model $P(U,G,I)$, each orthonormal (= ON) system in U is finite. Therefore U is locally sequentially compact, but there is no ON base for U. A similar situation holds for the Dworetzky-Rogers characterisation of finite dimensional spaces ([9], Theorem 1.c.2). But in combination we get the effective result:

1 Theorem *In ZF^o the following statements are equivalent for a Hilbert space H:*

(a) *The closed unit ball is sequentially compact.*
(b) *Each unconditionally convergent series converges absolutely.*
(c) *Each ON-system is Dedekind-finite.*

Proof: In ZF^o a first countable Hausdorff space X is sequentially compact, iff each closed, discrete set is D-finite. Though only the obvious part of this remark is used, a proof of its nontrivial implication can be given as follows:

Suppose X is not sequentially compact and $x:\underline{w} \rightarrow X$ is a sequence without any convergent subsequence. We shall show that $Im(x)$ is closed and discrete but not D-finite (Dedekind-finite). Because X is first countable, for $p \in Im(x)^-$, the closure of $Im(x)$, there is a neighborhood system $(U_k)_{k \in \underline{w}}$ and we set $n(0) = 0$, $n(k + 1) = \min N_k$ if $N_k = \{n > n(k):x_n \in U_k\} \neq \phi$. There is a k such that $N_k = \phi$. If not, the subsequence $(x(n(k)))_{k \in \underline{w}}$ of x converges to p. Hence

Received June 15, 1981; revised November 23, 1981