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A Note on Intuitionistic Models of ZF

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An analysis of the forcing construction of models of *ZF* by means of intuitionistic logic and Kripke semantics (or sheaf semantics) has been made by many authors (see for example [1], [2], [5]). We feel however that some basic information is still available from this approach, the motto of which could be "think of a sheaf of structures as *a* structure, using intuitionistic logic". In this note, we are not really interested in obtaining independence results, but want to sketch a set-theoretic forcing along lines which closely agree with that motto. In usual forcing constructions, there are two sheaf structures:

$$\langle M, - \epsilon_p - \rangle_{p \in C}$$

and

$$\langle M, p \Vdash - \epsilon - \rangle_{p \in C},$$

and passing from the first to the second is nothing else than a sheaf-version of the extensionalization of a binary relation. This is discussed in Section 2 for Kripke-structures but the proofs are designed so as to immediately extend to sheaf structures as is indicated in Section 4. Application to the construction of models of *ZF* is given in Section 3. To this we add definitions and notions which bring the construction closer to the classical construction of inner models of *ZF* and, as a "test-case", indicate how to adapt the well-known proof of relative consistency of $V \neq L$.

Roughly speaking, we may distinguish between two approaches to forcing. In the first, the model, be it Boolean-valued or intuitionistic, is constructed from the base model M as a union of a sequence of M_α indexed by ordinals of M ([1], [2], [4], [5]). In the second, no hierarchy is present, at least to begin with, and the hierarchy effect remains concentrated in the various forms of induction [7]. Our approach is of the second kind and may be viewed as an intuitionistic version of [7], whereas [2] for example is an intuitionistic construction of the first kind. In fact, this distinction is not so clear-cut and the

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