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Provable Fixed Points in $I\Delta_0 + \Omega_1$

ALESSANDRA CARBONE

Abstract It is shown that of the results of de Jongh–Montagna (Provable fixed points, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 1988) on provable fixed points in PA at least the positive part can be obtained for the system of bounded arithmetic of Wilkie and Paris (On the scheme of induction for bounded arithmetic formulas, *Annals of Pure and Applied Logic*, 1987). The methods used include use of a weakening of the scheme of Sigma-completeness due to A. Visser (this volume) which is valid for bounded arithmetic. The results imply that the results on shortenings of proofs due to Parikh (Existence and feasibility in arithmetic, *The Journal of Symbolic Logic*, 1971) apply to bounded arithmetic.

1 Introduction[1] This work should be considered as part of the general investigation into the arithmetical system $I\Delta_0 + \Omega_1$. We will present a refinement to $I\Delta_0 + \Omega_1$ of a result stated in de Jongh and Montagna [4] on witness comparison formulas having only provable fixed points in PA.

Briefly, let us introduce the arithmetical system and some of its properties: $I\Delta_0 + \Omega_1$ (cf. Paris and Wilkie [7]) is a set of axioms expressing the elementary arithmetic properties of the basic symbols 0, ′, +, *, ≤ (in the following we will refer to the obvious first-order language containing these symbols as S) together with the bounded induction schema $I\Delta_0$ (defined in S):

$$\forall x, z(\varphi(x,0) \wedge \forall y \leq z.(\varphi(x,y) \rightarrow \varphi(x,y')) \rightarrow \forall y \leq z\varphi(x,z)) \quad (\varphi \in \Delta_0)$$

plus the S-sentence Ω_1 expressing $\forall x \exists y.\omega_1(x) = y$, where $\omega_1(x) := x^{|x|}$ and $|-|$ is the length function for the binary representation of x.

We note that by the following result of Verbrugge [10]:

If NP $\neq$ CO-NP then

$$\nvdash_{I\Delta_0+\Omega_1} \forall b, c(\exists a(\mathrm{Prf}(a,c) \wedge \forall z \leq a\ \neg\mathrm{Prf}(z,b)) \rightarrow \mathrm{Pr}(\ulcorner \exists a \mathrm{Prf}(a,c) \wedge \forall z \leq a \neg \mathrm{Prf}(z,b)\urcorner))$$

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