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Notre Dame Journal of Formal Logic
Volume 32, Number 4, Fall 1991

On the Σ_1^0-Conservativity of Σ_1^0-Completeness

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Abstract In this paper we show that $I\Delta_0 + \Omega_1$ verifies the sentential Σ_1^0-conservativity of schematical, sentential Σ_1^0-completeness. (This means that for any finite set of Σ_1^0-sentences S we can prove in $I\Delta_0 + \Omega_1$ that the statement expressing the completeness of S w.r.t. $I\Delta_0 + \Omega_1$ is conservative over $I\Delta_0 + \Omega_1$ w.r.t. Σ_1^0-sentences.) Some consequences are discussed. We formulate a system of provability logic based on the verifiable sentential Σ_1^0-conservativity of schematical, sentential Σ_1^0-completeness.

1 Introduction As is well known it is a difficult question whether $I\Delta_0 + \Omega_1$ proves Σ_1^0-completeness. From Buss [1], Chapter 8, we can extract the following point: let $A(x)$ be any coNP-complete Π_1^b formula. Suppose $I\Delta_0 + \Omega_1$ proves: $\forall x(A(x) \rightarrow \Box_{I\Delta_0+\Omega_1}A(x))$. Then by Parikh's Theorem for some polynomial $P(x)$, $I\Delta_0 + \Omega_1$ proves: $\forall x(A(x) \rightarrow \exists |y| < P(|x|)\ \mathrm{Proof}_{I\Delta_0+\Omega_1}(y, A(x)))$. Hence *in the standard model* we have: $\forall x(A(x) \leftrightarrow \exists |y| < P(|x|)\ \mathrm{Proof}_{I\Delta_0+\Omega_1}(y, A(x)))$. In other words, $A(x)$ is equivalent to a Σ_1^b-predicate. Ergo NP = coNP. On the other hand, if $I\Delta_0 + \Omega_1$ *proves* a suitable schematic version of NP = coNP, then—as is easily seen—$I\Delta_0 + \Omega_1$ proves Σ_1^0-completeness.

Verbrugge [7] shows that for $A(x)$ in the above argument we may also take a formula of the form: $\Box_{I\Delta_0+\Omega_1}B(x) < \Box_{I\Delta_0+\Omega_1}C(x)$. Such a formula is $\exists\Pi_1^b$. This means that if completeness for Rosser-ordered provabilities (with parameter) were provable in $I\Delta_0 + \Omega_1$, then again NP = coNP.

In Paris and Wilkie [4] it is shown that all principles of Löb's Logic are valid in $I\Delta_0 + \Omega_1$. Solovay's proof of the arithmetical completeness of Löb's Logic, however, uses essentially the verifiability of schematical, sentential Σ_1^0-completeness (in fact: completeness for Rosser-ordered provabilities) in the arithmetical theory (see [7]). As a consequence, the question of arithmetical completeness of Löb's Logic for interpretations in $I\Delta_0 + \Omega_1$ is still open.

In this paper we show that for any finite set S, $I\Delta_0 + \Omega_1$ verifies that the statement expressing the completeness of S w.r.t. $I\Delta_0 + \Omega_1$ is conservative over $I\Delta_0 + \Omega_1$ w.r.t. Σ_1^0-sentences. In other words: $I\Delta_0 + \Omega_1$ verifies the sentential Σ_1^0-conservativity of schematical, sentential Σ_1^0-completeness over $I\Delta_0 + \Omega_1$. This

Received October 2, 1990; revised February 27, 1991