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Inequality in Constructive Mathematics

WIM RUITENBURG

Abstract We present difference relations as a natural generalization of inequality in constructive mathematics. Differences on a set S are defined as binary relations on all powers S^n simultaneously, satisfying axiom schemas generalizing the ones for inequality. The denial inequality and the apartness relation are special cases of a difference relation. Several theorems in constructive algebra are given that unify and generalize well-known results in constructive algebra previously employing special cases of difference relations. Finally, we discuss extended differences for a set S as collections of relations defined on all powers S^X simultaneously.

Introduction In mathematics the natural generalization of equality is equivalence. A theory with equivalence involves the reflexive, symmetric, and transitive equivalence, and functions and relations respecting this equivalence. In constructive mathematics the same theory with equivalence relations works without difficulty. For inequality the situation is more complicated. There are different versions of constructive inequality that only in classical mathematics are equal to the one standard inequality. Examples are: denial inequality, where $x \neq y$ if and only if it is not true that $x = y$, that is, $\neg x = y$; and tight apartness, whose axiomatization we will present later on. The natural inequality on the set of real numbers $\mathbf{R}$, defined by $r \neq s$ if and only if $|r - s| > 1/n$ for some natural number n, is a tight apartness. Tight apartness and denial inequality are independent; a tight apartness need not be a denial inequality, a denial inequality need not be a tight apartness. We know of no definition of a binary relation on a set S, generalizing both denial inequality and apartness, that allows for a substantial constructive theory of inequality.

There are several theorems in algebra and elsewhere that hold if we use denial inequality as the intended inequality, and that also hold if we use a tight apartness as the intended inequality. Sometimes there may even be a third version of inequality that makes the theorem work. For each of these cases we need a new proof to establish our result. For a uniform treatment of such theorems we

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