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Incompleteness in Intuitionistic Metamathematics

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Abstract There are three main results; all are contributions to the intuitionistic metatheory of intuitionistic systems. First, pure intuitionistic predicate logic is provably incomplete with respect to ordinary model-theoretic semantics, provided that the metatheory is suitably intuitionistic. With the same proviso, intuitionistic propositional logic is also incomplete; in fact, the concept of validity for formulas in one propositional variable is not arithmetically definable. Also, one cannot prove—in standard intuitionistic metatheories—an existence theorem for countable models, even when the relevant theory is that of subfinite sets in the language of pure identity.

1 Introduction The three main results are all contributions to the intuitionistic metatheory of intuitionistic systems. In more detail, the first theorem shows the intuitionistic predicate logic to be, in the presence of a weak form of Church's Thesis (WCT), neither complete nor almost complete. The proof itself is fully constructive: there is a single formula ϕ such that Con(ϕ) is provable in a fragment of Heyting arithmetic while WCT implies that ϕ has no model. Although the conclusion of the proof, that intuitionistic predicate logic is incomplete, is not especially new, it seems that the proof given here is simpler and more straightforward than existing published proofs of similar results. Our proof joins the *incompleteness of intuitionistic logic* directly to standard classical proofs of the *incompleteness of first-order arithmetic*. Basically, WCT blocks the construction of classically possible structures $\mathfrak{A}$ because, in those structures, truth in $\mathfrak{A}$ would be definable—in contradiction to the Gödel-Tarski fixed-point theorem, which is correct both classically and constructively.

The second result, on intuitionistic propositional logic, shows that completeness theorems for propositional logic are almost as elusive as those for predicate logic. In the proof, the axioms of intuitionistic set theory together with Church's

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