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Bounds in Weak Truth-Table Reducibility

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Abstract A necessary and sufficient condition on a recursive function is given so that arbitrary sets can be truth-table reduced via this function as the bound. A corresponding hierarchy of recursive functions is introduced and some partial results and an open problem are formulated.

Weak truth-table reducibility, often called bounded Turing reducibility, is defined as follows: $A \subseteq \omega$ is weak-truth–table reducible to $B \subseteq \omega$ ($A \leq_{\mathrm{wtt}} B$) if there is a recursive function f and an algorithm which answers questions of the form "$n \in A$?" when supplied answers to any questions it asks of the form "$m \in B$?" for $m \leq f(n)$. The function f is called the bound of the reduction.

The hierarchy of subsets of ω induced by the relation $\leq_{\mathrm{wtt}}$ was extensively studied in the past (cf. [1]). In this paper, however, a hierarchy of the *bounds* (i.e. of recursive functions) is considered. We denote by $\mathcal{S}(f)$ the set of A such that there is a B such that $A \leq_{\mathrm{wtt}} B$ via a reduction with bound f, and we write $f \ll g$ iff $\mathcal{S}(f) \subseteq \mathcal{S}(g)$. Of course, $\mathcal{R} \subseteq \mathcal{S}(f) \subseteq 2^{\omega}$ for all recursive functions f. We give necessary and sufficient conditions on f for $\mathcal{S}(f) = \mathcal{R}$ and for $\mathcal{S}(f) = 2^{\omega}$, i.e. for f being on the bottom and on the top of the hierarchy induced by $\ll$. We also give a necessary condition for $f \ll g$.

Our interest is focused to the bound of the wtt-reduction by the following phenomenon: A part of an (in general) nonrecursive set B can be given by a list. Having the set A Turing reduced to B, a part of A is given which corresponds to the list of a part of B, and which may be much larger than the list itself—depending mainly on the bound of the reduction.

A motivation for the study of our hierarchy of bounds comes also from the theory of nets of automata. Consider a chain of automata numbered by natural numbers. Suppose each automaton is in one of the two states 0 and 1. Then the state of the whole net is uniquely determined by a set $B \subseteq \omega$ in an obvious way. Now let the automata work, and after some time all of them may stop and the net may come into a state determined by a set A. In a fairly devised net we would have $A \leq_{\mathrm{wtt}} B$. The bound f of this reduction depends on how the

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