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Enumerations of Turing Ideals with Applications

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Abstract We examine enumerations of ideals in the Turing degrees and give several applications to the model theory of first- and second-order arithmetic.

A *Turing ideal* is a collection of subsets of ω closed under Turing reducibility and join. If I is a countable Turing ideal we say that E is an *enumeration* of I if and only if $I = \{E_n : n \in \omega\}$ where $E_n = \{m : \langle n,m\rangle \in E\}$. Enumerations of Turing ideals play an important role in the study of degrees coding recursively saturated models of Peano Arithmetic (see [5]). Our goal in this paper is to point out some simple facts about enumerations of Turing ideals and examine their consequences in the model theory of first- and second-order arithmetic.

1 ω-incompleteness theorems We consider three subsystems of second-order arithmetic, RCA_0, ACA_0, and WKL_0. RCA_0 is axiomatized by P^- (Peano Arithmetic without the induction axioms), the axiom of extensionality and the schemas of Σ_1^0-induction and recursive comprehension. WKL_0 is obtained from RCA_0 by adding an axiom saying every infinite subtree of $2^{<\omega}$ has an infinite path. ACA_0 is obtained from RCA_0 by adding the schema of arithmetic comprehension. For further information on these theories the reader should consult [7].

Our recent interest in this subject was motivated by considering the following incompleteness theorem of Steel.

Theorem 1.1 (Steel [9]) *Let T be an ω-consistent arithmetic extension of ACA_0. There is an ω-model* $\mathbf{M}$ *of T such that* $\mathbf{M} \models$ *"there is no ω-model of T".*

Thus even in ω-logic T does not prove its own ω-consistency. Steel's result is actually much stronger. Suppose $\mathbf{M} = (\omega, X)$ and $\mathbf{N} = (\omega, Y)$ are models of RCA_0. We say that $\mathbf{M} \gg \mathbf{N}$ if and only if there is $E \in X$ an enumeration of Y. If $\mathbf{M} \models$ "there is an ω-model of T", then there is an $E \in X$ such that $\mathbf{N} =$

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