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Provable Forms of Martin's Axiom

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Abstract It is shown that if Martin's Axiom (MA) is restricted to well-orderable sets, and if the countable antichain condition is replaced by either a finite antichain condition or a finite "Q-strong antichain" condition, then the resulting statements are forms of MA provable in ZF. Variations of these weak forms of MA are shown to be equivalent to the Axiom of Choice and some of its weak forms.

Introduction The purpose of this paper is to set forth the proof in ZF of some weak forms of Martin's Axiom (MA). This may be of some philosophical interest with respect to the work of Maddy ([7],[8]). To obtain such forms: (i) MA is restricted to quasi-orders on well-orderable sets, and (ii) the countable antichain condition is replaced by a finite antichain condition. The conclusion of MA is also strengthened, for under these conditions there exist filters which intersect all dense subsets. These theorems are applied to prove MA-type equivalents of forms of the Axiom of Choice (AC).

One known result related to this paper is the statement that, in ZF, $AC^{\aleph_0}$ implies MA ($\aleph_0$) (Kunen [6], Lemma 2.6(c), pp. 54–55). Another related result (due to Goldblatt [3]) is that, "In ZF, the principle of dependent choices is equivalent to the statement that if $(P,\leq)$ is a nonempty partial order and $\mathfrak{D}$ is a countable collection of dense subsets of P, then there exists a $\mathfrak{D}$-generic filter".

Definitions Let $(P,\leq)$ be a quasi-order, i.e., $\leq$ is reflexive and transitive (if in addition $\leq$ is antisymmetric, then $(P,\leq)$ is a partial order).

Two elements x,y of P are said to be *incompatible* if there does not exist $z \in P$ such that $z \leq x$ and $z \leq y$ (otherwise x and y are said to be *compatible*).

A subset I of P is said to be an *antichain* if any two elements of I are incompatible. $c(x)$ denotes the set of elements of P that are compatible with x.

A subset D of P is said to be *dense* if for all $x \in P$ there exists $y \in D$ such that $y \leq x$.

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