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Cofinalities of Countable Ultraproducts: The Existence Theorem

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Abstract We show that there exists an ultrafilter U in the set $\mathbb{N}$ of natural numbers such that the cofinality of the U-ultrapower ${}^{\mathbb{N}}\mathbb{N}/U$ equals $\mathrm{cof}(d)$, where d is the minimal cardinality of a dominating subfamily of ${}^{\mathbb{N}}\mathbb{N}$. Moreover, the coinitiality of the family of finite-to-one functions in this ultrapower is also $\mathrm{cof}(d)$. If $c = d$, then U may be taken to be a P-point.

All reals are understood to be in ${}^{\omega}\omega$. EAL is the family of *eventually arbitrarily large* reals (also known as finite-to-one reals). If f, g are two reals, we say that *f majorizes g* and *g minorizes f* if $(\exists m \forall n > m) f(n) > g(n)$. Every real may be majorized by a nondecreasing real, and every EAL real may be minorized by a nondecreasing EAL real. A family of reals D is said to be a *majorizing* family if every real is majorized by a real from D. An EAL family of reals D' is said to be a *minorizing* family if every EAL real is minorized by a real from D'. D is said to be *unbounded* if there is no real which majorizes every real in D. Following [6], we let d be the minimal cardinality of a majorizing family of ${}^{\omega}\omega$ and let b be the minimal cardinality of an unbounded family. The cardinal b must be regular and uncountable, but it is consistent that d be singular. However, $\mathrm{cof}(d) \geq b$. (These and other interesting properties of these cardinals are discussed in [7] and [8].) It is easy to check that d is also the minimal cardinality of a minorizing family of EAL. (Proofs may be found in [3] and [4].)

All filters and ultrafilters are understood to be proper, nonprincipal, and on ω. If U is such an ultrafilter, then U induces a linear-ordering $<_U$ on ${}^{\omega}\omega$ where $f <_U g$ iff $\{n \mid f(n) < g(n)\} \in U$. We will write ${}^{\omega}\omega/U$ (respectively EAL$/U$) for the structure consisting of (U-equivalence classes of) reals (respectively EAL reals) together with this ordering. In this paper we will consider the possible cofinalities and coinitialities of EAL$/U$. (EAL is a final segment of ${}^{\omega}\omega$; hence the cofinality of ${}^{\omega}\omega/U$ equals that of EAL$/U$.) In the terminology of [10], these are precisely the cofinalities and coinitialities of skies in nonstandard

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