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Peano's Smart Children: A Provability Logical Study of Systems with Built-in Consistency

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Abstract The systems studied in this article prove the same theorems (from the "extensional" point of view) as Peano Arithmetic, but are equipped with a self-correction procedure. These systems prove their own consistency and thus escape Gödel's second theorem. Here, the provability logics of these systems are studied. An application of the results obtained turns out to be the solution to a problem of Orey on relative interpretability.

1 Introduction Consistency can be built into a system in various ways. The two best known constructions are Rosser's and Feferman's, both of which take a given formal system in the usual sense as initial data. Consider, for example, Peano Arithmetic (PA). A proof in the Peano System will count as a proof in the Rosser System based on PA, if there is no shorter Peano proof of the negation of its conclusion. The Feferman System can be described in various interesting ways, modulo provable equivalence in PA of the formulas defining the set of theorems. One such way is this: A proof in the Peano System will count as a proof in the Feferman System based on PA, if the finite set of arithmetical Peano axioms smaller than or equal to the largest arithmetical Peano axiom used in the proof is consistent.

The reasons such constructions occur in the literature are various:

(i) They serve as counterexamples in the study of the relations between Gödel's first and second Incompleteness Theorems (see [4]).
(ii) They serve as didactic examples in philosophical discussions, like the

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