Notre Dame Journal of Formal Logic
Volume 34, Number 1, Winter 1993

The Amalgamation Property in Normal Open Induction

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Abstract It is known that open induction (OI), the fragment of Peano arithmetic, fails to have the joint embedding property, a result due to Wilkie. On the other hand we have proved that if we require our models to be normal, that is, to be integrally closed in their fraction fields, the corresponding theory NOI extending OI, has the joint embedding property. Here we prove NOI does not have the amalgamation property.

1 Introduction Let $\mathcal{L}$ denote the first-order language of ordered rings based on the symbols $0, 1, +, -, \cdot, <$. *Open Induction* (OI for short) is the fragment of Peano Arithmetic axiomatized by the axioms for discretely ordered rings together with the following axiom-scheme:

$$\forall \bar{x}((\theta(\bar{x},0) \wedge \forall y \geq 0(\theta(\bar{x},y) \rightarrow \theta(\bar{x},y+1)) \rightarrow \forall y \geq 0 \theta(\bar{x},y)),$$

where $\theta(\bar{x},y)$ is a quantifier-free $\mathcal{L}$-formula.

In this article rings are always commutative with 1.

A first point in the study of models of OI is the following algebraic characterization.

Theorem 1.1 (Shepherdson) *Let M be a discretely ordered ring. M is a model of open induction if and only if for all r in the real closure of the fraction field of M there is an $a \in M$ such that $|r-a| < 1$.*

Shepherdson [6] proved several independence results for OI using this criterion. In particular, he showed that OI does not prove the normality axioms. We remind the reader that a domain is called normal if it is integrally closed in its fraction field.

Let NOI denote the extension of OI by adding the normality axioms, that is, for each $n \in \mathbf{N}^*$

$$\forall \bar{z}, x, y(x, y \neq 0 \wedge x^n + z_1 x^{n-1} y + \cdots + z_n y^n = 0 \rightarrow \exists w(x = wy)).$$

Received July 23, 1992; revised December 14, 1992