Notre Dame Journal of Formal Logic
Volume 33, Number 3, Summer 1992

The Categoricity Spectrum of Pseudo-elementary Classes

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Abstract Given a pseudo-elementary class $\mathcal{K}$ we investigate the associated class of cardinals where $\mathcal{K}$ is categorical. We show that any such class must be closed and if it is nonempty then there is an ordinal $\delta \neq 0$ so that $\{\kappa : \kappa < \beth_{\delta \cdot \alpha}$ and $\mathcal{K}$ is κ-categorical$\}$ is closed and unbounded in $\beth_{\delta \cdot \alpha}$ for all $\alpha > 0$. Also, assuming the consistency of a huge cardinal, we show that the statement "$\mathcal{K}$ $\aleph_2$-categorical implies $\mathcal{K}$ $\aleph_3$-categorical" is independent of ZFC.

1 Introduction In [8] Morley proved that if an elementary class in a countable language is categorical in some uncountable power, then it is categorical in all uncountable powers. The result was extended to elementary classes in uncountable languages by Shelah. The aim of this paper is to explore possible generalizations of these results to pseudo-elementary (PC_Δ) classes (i.e., reducts of an elementary class to a smaller language).

In [1] Keisler proved that one direction of Morley's theorem extends to pseudo-elementary classes. He showed that if a PC_Δ class in a countable language is $\aleph_1$-categorical, then it is categorical in every uncountable power. However, Silver gave an example of a PC_Δ class that is κ-categorical if and only if κ is a strong limit cardinal. The other known positive result was proved independently by Keisler [4], Čudnovskiĭ [2], and Shelah [11]. Suppose $\mathcal{K}$ is a PC_Δ class whose underlying language has power λ and $\beth_\delta$ is the Hanf number for omitting a type in a first order language of power λ (e.g., if $\lambda = \aleph_0$ then $\beth_\delta = \beth_{\omega_1}$). They showed that if $\mathcal{K}$ is categorical in some power $> \lambda$, then $\mathcal{K}$ is categorical in all powers $\beth_{\delta \cdot \alpha}$ for $\alpha \in \mathrm{ORD}$, $\alpha > 0$.

In Section 2, we obtain two new positive results. First, we show that for any pseudo-elementary class $\mathcal{K}$, the class of cardinals κ where $\mathcal{K}$ is κ-categorical is closed in the order topology. Next we extend the result above by showing that if $\mathcal{K}$ is a pseudo-elementary class whose underlying language has power λ and $\mathcal{K}$ is categorical in some power $> \lambda$ then $\{\beta < \delta \cdot \alpha : \mathcal{K}$ is $\beth_\beta$-categorical$\}$ is closed and unbounded in $\delta \cdot \alpha$ for all $\alpha > 0$. In particular, if the underlying language of $\mathcal{K}$ is countable and $\mathcal{K}$ is categorical in some uncountable power, then $|\{\mu <$

Received March 29, 1991