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A Diophantine Definition of Rational Integers over Some Rings of Algebraic Numbers

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Abstract The author considers the rings of algebraic numbers integral at all but finitely many primes in the number fields, where it has been previously shown that Hilbert's Tenth Problem is undecidable in the rings of algebraic integers, and proves that the problem is still undecidable in the bigger rings by constructing a diophantine definition of rational integers there.

1 Introduction Hilbert's Tenth Problem can be phrased as the following question. Is there an algorithm to determine, given an integer polynomial equation $f(x_1, \ldots, x_n) = 0$, whether this equation has integer solutions? This question was answered negatively by Davis, Robinson, Matijasevich, and Putnam. (See Davis et al. [2] and Davis [1].) One of the major and still unresolved problems in the area is the same question applied to rings of algebraic integers of a general number field as well as number field itself. The problem is also still unresolved for $\mathbb{Q}$.

The present paper can be a step in the direction of resolving the problem for some number fields. Instead of the rings of algebraic integers, the author considers the Diophantine problem over the rings of algebraic numbers where finitely many primes are allowed to appear in the denominators. Using the Pell equation technique similar to the one introduced in a proof of the original problem (see [1]) and extended by Denef in [5], the author shows that in all the fields where Hilbert's Tenth Problem is known to have no solution in the rings of algebraic integers, with the exception of the case of the extensions of degree 4 with no real subfield, the problem is still unsolvable in the bigger rings described above.

Besides the ring of rational integers, the Diophantine problem is known to be undecidable in the rings of algebraic integers of all the totally real fields, fields of degree 2 over totally real fields, fields with one pair of complex conjugate embeddings, fields of degree 4 with a subfield of degree 2, and all the subfields of the above-mentioned fields. These subfields include all the abelian extensions

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