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A Parity-Based Frege Proof for the Symmetric Pigeonhole Principle

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Abstract Sam Buss produced the first polynomial size Frege proof of the pigeonhole principle. We introduce a variation of that problem and produce a simpler proof based on parity. The proof appearing here has an upper bound that is quadratic in the size of the input formula.

1 Introduction The proof complexity of the pigeonhole principle is a well studied problem. In 1985, Haken [6] proved that any resolution proof for it must have exponential size. In 1987, Buss [4] produced a polynomial size Frege proof. Ajtai [1] first showed that any polynomial Frege proof must have greater than constant depth. Most recently, Beame, Impagliazzo, Krajíček, Pitassi, Pudlák, and Woods [3] showed an $\Omega(\log \log n)$ lower bound on the depth of any polynomial Frege proof.

Buss's Frege proof proceeds by building up predicates to count the number of atoms set to true, i.e., a propositional multiplexer. With the binary results, the proof then introduces predicates to compute "less than." Starting with the constraints that each pigeon must be placed in some hole, the proof derives that n is less than the number of holes filled by the first $n + 1$ pigeons. Using the constraints that each hole may contain at most one pigeon, the proof derives that the number of holes filled by the first $n + 1$ pigeons is less than or equal to n. In this way, Buss produces a contradiction. He estimates the size of the proof to be $O(n^{20})$.

The main result of this paper is a proof similar in structure to Buss's proof. We introduce a variation of the pigeonhole principle which is referred to as the symmetric pigeonhole principle. With this variation it is possible to get a simpler Frege proof that is based on parity rather than less than.

The symmetric pigeonhole principle is formalized as follows. For both problems, begin with the same variable space P_{ij}, $1 \leq i \leq (n + 1)$, $1 \leq j \leq n$, where P_{ij} is true iff pigeon i is placed in hole j. The traditional pigeonhole principle, abbreviated PHP_n, is the conjunction of the following two constraints.

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