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# ON NEW THEOREMS FOR ELEMENTARY NUMBER THEORY

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*Introduction*: This paper* generalizes the classical number-theoretic notions of multiplicative function and additive function. In addition to extending classes of functions it can be shown that certain specific functions such as, e.g., Euler's totient, Möbius' function, and Liouville's function have precise analogues within the theory. Results of G. H. Hardy and S. Ramanujan are used in the study.

1. Apply the Unique Factorization Theorem (UFT) of [1], *viz.*, $n = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$ ($p_i$ simply ordered, and distinct) to its own natural number exponents $>1$, and apply the UFT to the exponents $>1$ so obtained, etc. (use induction), until the process terminates, for a given $n$, with a *unique* "constellation" of prime numbers alone called a *mosaic* [2;3]. E.g., *the* mosaic of 10,000 is $2^{2^2} \cdot 5^{2^2}$. This process provides a simple revision of the standard Gaussian model of the UFT [3].

*Definition 1.* A number-theoretic function $f$ is said to be *generalized multiplicative* provided $f(a \cdot b) = f(a) \cdot f(b)$, if the mosaics of $a$ and $b$ have no prime in common.

*Lemma 1. Every standard multiplicative number-theoretic function* [1] *is generalized multiplicative, and there exists a generalized multiplicative function* $\psi^2$ *(defined as* $\psi(\psi(\cdot))$*, where* $\psi$ *is defined as follows:* $\psi(n)$ *is the product of the primes alone in the mosaic of n which is not multiplicative; i.e., the class of generalized multiplicative functions property contains the class of multiplicative functions.*

*Proof.* Clearly every multiplicative function is generalized multiplicative, since if the function "factors" when $a$ and $b$ have no prime base in common (but possibly their mosaics *may* have some prime in common) then, *a fortiori*, the function "factors" when the mosaics of $a$ and $b$ have no prime in common. Now, since $\psi$ is multiplicative $\psi^2(a \cdot b) = \psi(\psi(a) \cdot \psi(b))$. But, if the mosaics of $a$ and $b$ have no prime in common then $(\psi(a), \psi(b)) = 1$.

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