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A THEOREM ON S4.2 AND S4.4

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Theorem. If ML is abbreviated as R, the pure C-N-R-fragment of S4.2 can be axiomatized and contains a model of S4.4.

Proof. (i) The following theses and rule are in S4.2:

R1. $CRCpqCRpRq$
R2. $CRpRRp$
R3. $CNRpRNRp$
R4. $CRNpNRp$
R5. *From* α *to infer* $R\alpha$

Indeed all but *R1*, *R4* are in S4. Let **PC**, C-detachment, substitution, *R1*-*R5* be denoted as $\{R\}$. Taking $\{R\}$ as primitive and the definition

Df.L $L\alpha = K\alpha R\alpha$

we can obtain the theses and rule

L1. $CLpp$
L2. $CLpLLp$
L3. $CpCNLpLNLp$
L4. *From* α *to infer* $L\alpha$
L5. $CLCpqCLpLq$
L6. $CNLNLpRp$
L7. $CRpNLNLp$.

L1-*L5* constitute a model of S4.4.

(ii) $\{R\}$ is complete for pure C-N-R-theses in S4.2. For let α be such a thesis; then there is a corresponding ML-thesis provable from **PC**, *L1*-*L5*, since S4.4 contains S4.2. But then by *L6*, *L7* the R-thesis is provable in the L-system, and so from $\{R\}$. (i) and (ii) prove the theorem. It follows that the matrix of S4.2 can be used to decide S4.4—just eliminate L in the expression under test, by Df.L, and see whether the result is provable in S4.2.

It is worth noting that *L1*-*L6* follow from *R1*-*R3* and *R5*. But *R4* is independent (take R as Verum) and is needed for *L7*.

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