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ON THREE RELATED EXTENSIONS OF S4

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1. This paper contains three algebraic exercises which I was led to by some recent discussions with Mr. Geach. The first is to provide a proof of decidability and a characteristic model for the system obtained by extending S4[1] with

$$CMLpCpLp \quad ,$$

a system originally studied by Geach, and later by various logicians. The others arise from an enquiry of Geach's into systems which can be obtained by extending S4 with words in a single variable. Could, for example, S4.3 be so presented? It could not, but I was able to establish that its strongest fragment of this form was S4.2 + *ALCLCpLpLpLCLCLCpLpLpLp*, and that its weakest extension of this form was S4 + *CMLpALCpLpLCLCLCpLpLp*. Geach then pointed out that extending these systems with *CMLpCLCLCpLpLpLp* gave the trio S4.2 + *CMLpCLCLCpLpLpLp*, D,[2] S4 + *CMLpCpLp* (in order of increasing strength). As it seems to me that this is a more interesting trio than mine, I have adjusted my techniques to apply to it instead. My second exercise, then, is to show that S4 + *CMLpCpLp* stands to D as the weakest extension of S4 with single-variable axioms containing it; my third is to show that the closure in S4 of the single-variables theses of D is S4.2 + *CMLpCLCLCpLpLpLp*.

I use the usual machinery of closure algebras, the order closure models of [2], and the finite model property. My main concern is with the well-connected closure algebras—i.e. those were

$$\mathbf{C}a \cap \mathbf{C}b = \wedge \text{ iff } a = \wedge \text{ or } b = \wedge$$

which are known to characterize all (normal) extensions of S4, by Lemma 3 of [1].[3] I use the symbol - for relative complement, rather than in its normal role of complement proper. Otherwise, anything I use here will be found in [1] or [2], or in the papers referred to there.

2. The system S4 + *CMLpCpLp* can also be axiomatised, more conveniently for my purposes, as S4 + *ALNLpLCpLp*. This new axiom is

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