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PRIMITIVE RECURSIVE COMPUTATIONS

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1. *Definition of a computation.** Using the definition of primitive recursive function found in Kleene, p. 219 [1], we shall define a (primitive-recursive) computation, investigate the mechanics of executing such a computation, and derive upper bounds for the value of the function and for the number of steps required for the computation.

Kleene's definition is: "Each of the following equations and systems of equations (I)-(V) defines a number-theoretic function ϕ, when n and m are positive integers, i is an integer such that $1 \leqslant i \leqslant n$, q is a natural number, and $\psi, \chi_1, \ldots, \chi_m\chi$ are given number-theoretic functions of the indicated numbers of variables.

(I) $\phi(x) = x'$.

(II) $\phi(x_1, \ldots, x_n) = q$.

(III) $\phi(x_1, \ldots, x_n) = x_i$.

(IV) $\phi(x_1, \ldots, x_n) = \psi(\chi_1(x_1, \ldots, x_n), \ldots, \chi_m(x_1, \ldots, x_n))$.

(Va) $\begin{cases} \phi(0) = q, \\ \phi(y') = \chi(y, \phi(y)). \end{cases}$

(Vb) $\begin{cases} \phi(0, x_2, \ldots, x_n) = \psi(x_2, \ldots, x_n), \\ \phi(y', x_2, \ldots, x_n) = \chi(y, \phi(y, x_2, \ldots, x_n), x_2, \ldots, x_n). \end{cases}$

((Va) constitutes the case of (V) for $n = 1$, and (Vb) for $n > 1$.) A function is *primitive recursive* if it is definable by a series of applications of these five operations of definition."

Modifying this definition to permit zero arguments in (II) so that (Va) and (Vb) can be combined, we proceed in the obvious way to give a recursive definition of *function word*, giving in the process a definition of the *rank* of a function word:

(1) S is a function word of rank 1.
(2) C_m^n is a function word of rank n ($n \geqslant 0$).
(3) U_m^n is a function word of rank n ($n \geqslant 1$; $1 \leqslant m \leqslant n$).
(4) If A^m is a function word of rank m and if $B_1^n, \ldots, B_m^n$ are function words of rank n, then $\mathsf{S}_m^n A^m B_1^n \ldots B_m^n$ is a function word of rank n ($m \geqslant 1$, $n \geqslant 1$).

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