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AN ELEMENTARY CONSTRUCTION OF THE NATURAL NUMBERS

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I. *Introduction* This paper presents a set-theoretic construction of the natural numbers which employs, besides standard set-theoretic operations, only The Axiom of Choice and the existence of an infinite set.

The following notation will be used. A set of sets will be called a *family*. Set-theoretic inclusion will be represented by $\subseteq$, and strict inclusion by $\subset$. The power-set of a set S will be represented by $\mathsf{P}(S)$. If f is a function defined on a set S, then $f(T)$ will represent the set of images under f of the elements of T, for each subset T of S. In particular $f(\phi) = \phi$, where ϕ is the void set. If $\mathscr{A}$ is a family of subsets of a set S, then $\bigcap \mathscr{A}$ will represent the intersection of the members of $\mathscr{A}$. In particular, $\bigcap \phi = \phi$. The difference of sets S and T will be represented by $S \backslash T$.

The following definitions would be used. The pair $<S,g>$, where S is a set and g is a function on S, is called a *Peano System* if the following three conditions are satisfied:

(i) g is one-to-one,
(ii) g does not map S into S,
(iii) If T is a subset of S such that
$T \cap [S\backslash g(S)] \neq \phi$ and $g(T) \subseteq T$, then $T = S$.

We wish to construct a Peano System.

A *choice function* on a set S is a function which assigns to each non-void subset T of S an element of T. The *Axiom of Choice* states that a choice function may be defined on any set.

A set S will be called *finite* if and only if every one-to-one mapping in S maps S onto S. S will be called *infinite* if it is not finite. The following propositions give some properties of finite sets which will be used in the sequel.

Proposition 1: *If S is a finite set and T is a subset of S, then T is finite.*

Proof: If f is a one-to-one function in T, then the mapping g in S defined by:

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