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A THEOREM FOR DERIVING CONSEQUENCES OF THE AXIOM OF CHOICE

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I. *Introduction* This paper is addressed to the problem of proving results directly from the Axiom of Choice. A general theorem on mappings in partially-ordered sets will be proved, and proofs of Zorn's Lemma and the Well-Ordering Theorem will be given as corollaries to this theorem. The following concepts will be used.

A *partially-ordered set* is a set on which is defined a reflexive, transitive, anti-symmetric binary relation $\leqslant$. A *chain* is a totally-ordered subset of a partially-ordered set. If a partially-ordered set has a smallest and/or a greatest element, these will be represented respectively by 0 and 1. The least-upper-bound, if it has one, of a subset T of a partially-ordered set will be represented by $\bigsqcup T$. It should be noted that if every subset of a partially-ordered set X has a least-upper-bound, then $0 = \bigsqcup \phi$ and $1 = \bigsqcup X$ are in X, where ϕ is the void set.

A *choice function* on a set S is a function which assigns to each non-void subset T of S an element of T. The *Axiom of Choice* states that a choice function may be defined on any set.

The following additional notation will be employed. Set-theoretic inclusion will be represented by $\subseteq$, and strict inclusion by $\subset$. The power-set of a set S will be represented by $\mathsf{P}(S)$. If f is a function defined on a set S, then $f(T)$ will represent the set of images under f of the elements of T, for each subset T of S. In particular, $f(\phi) = \phi$. If $\mathscr{A}$ is a family of subsets of a set S, then $\bigcup \mathscr{A}$ and $\bigcap \mathscr{A}$ will represent respectively the union and intersection of the members of $\mathscr{A}$. In particular, $\bigcup \phi = \phi$ and $\bigcap \phi = S$. Finally, the difference of sets S and T will be represented by $S \backslash T$.

II. *The Main Theorem*

Theorem 1: *If X is a partially-ordered set in which each subset has a least-upper-bound, and g is a function from X into X which satisfies the following condition*:

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