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A Smart Child of Peano's

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Abstract We calculate the provability logic of a special form of the Feferman provability predicate together with the usual provability predicate of Peano Arithmetic. In other words, we construct a bimodal system with the intended interpretation of the expression $\Box\varphi$ being as usual the formalization of "φ is provable in PA" and the new modal operator $\triangle$ standing, when applied to φ, for "there exists an x s.t. $I\Sigma_x$ is consistent and proves φ". The new system is called **LF**. We construct a Kripke semantics for **LF** and prove the arithmetical completeness theorem for this system. A small number of other issues concerning the Feferman predicate, such as uniqueness of gödel-sentences for $\triangle$, is also considered.

1 Introduction The Feferman provability predicate for reflexive recursively axiomatized theories emerged for the first time in Feferman's paper [2]. Starting with a reflexive theory, Peano Arithmetic PA being the conventional example, one chooses a sequence of finitely axiomatized theories $(\mathrm{PA}\lceil n)_{n\in\omega}$ with $\mathrm{PA}\lceil n+1$ extending $\mathrm{PA}\lceil n$, and $\mathrm{PA} = \bigcup_{n\in\omega} \mathrm{PA}\lceil n$.

Outside modal-logical contexts, let us write $\triangle$ for the Feferman predicate reserving the shorthand $\Box$ for the usual provability predicate. $\triangle\varphi$ is then defined as the formalization of

"there exists an $x \in \omega$ s.t. $\mathrm{PA}\lceil x$ is consistent and $\mathrm{PA}\lceil x \vdash \varphi$".

The sequence $(\mathrm{PA}\lceil n)_{n\in\omega}$ is called the *base sequence* for this $\triangle$.

The *reflexivity property* of PA translates as saying that for all $n \in \omega$ PA proves that $\mathrm{PA}\lceil n$ is consistent. This was first established by Mostowski [13] and is crucial for practically all applications of $\triangle$.

The first use of $\triangle$ was to illustrate the relevance of the Hilbert-Bernays derivability conditions to Gödel's Second Incompleteness Theorem. The close connection of $\triangle$ to relative interpretability became apparent in Feferman [2], Orey

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