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THE MENGER ALGEBRAS OF 2-PLACE FUNCTIONS IN THE 2-VALUED LOGIC

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A Menger algebra of 2-place functions over a set Δ is (cf. [5]) a set $\mathfrak{S}$ of functions mapping $\Delta \times \Delta$ into Δ, which is closed with respect to substitution or composition, i.e., has the property that, for any three functions F, G, H belonging to $\mathfrak{S}$, the composite function $F(G,H)$ belongs to $\mathfrak{S}$. Here, $F(G,H)$ is the function assuming the value $F(G(x,y), H(x,y))$ for each (x,y) in $\Delta \times \Delta$. The purpose of this paper is to list all the Menger algebras of 2-place functions over $\{0,1\}$. The correspondence between these functions and the binary operators of the 2-valued logic is obvious.

We denote the 16 functions (cf. [1], [2]) by

	A	B	C	D	E	I	J	1	A'	B'	C'	D'	E'	I'	J'	0
(0,0)	1	0	1	1	1	0	0	1	0	1	0	0	0	1	1	0
(0,1)	1	1	1	0	0	0	1	1	0	0	0	1	1	1	0	0
(1,0)	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0
(1,1)	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

The smallest Menger algebra containing, as a subset, the set of functions $\{F_1, \ldots, F_k\}$ is said to be *generated*[1] by $\{F_1, \ldots, F_k\}$ (cf. [1] and [3]) and is denoted by $[F_1, \ldots, F_k]$. If $[F] = \{F\}$, or, which amounts to the same, if $F(F,F) = F$, then F is called *idempotent* (cf. [1]). We denote the set of all 16 functions as well as the algebra consisting of these functions by $\mathfrak{F}$. The set of the idempotent functions, that is $\{1,0,I,J,A',B\}$, is denoted by $\mathfrak{F}_0$. We further introduce an operator ν in $\mathfrak{F}$ by defining νF as the function whose value for each (x,y) in $\{0,1\} \times \{0,1\}$ is 0 or 1 according as $F(x,y)$ is 1 or 0. An algebra is said to be ν-closed if, whenever F is in it, νF is also.

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1. It should be remembered that in this purely algebraic sense of generation, the function A (corresponding to Sheffer's stroke) generates only 4 of the 16 functions in $\mathfrak{F}$ (cf. [1]).

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