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A NOTE ON HALLDÉN-INCOMPLETENESS

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In [3], Halldén in effect suggests that a modal (or other) system S [1] is unreasonable if there are wffs A and B, each containing one propositional variable and such that the variable in A is different from the variable in B, with the property that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$. In the same paper, Halldén shows that any system intermediate between the Lewis systems S1 and S3 is unreasonable in this sense. McKinsey [9] relaxes the condition that the wffs A and B contain just one variable; following this approach, let us say that a system S is *Halldén-incomplete* (H-incomplete) iff there are wffs A and B with no variables in common such that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$, and *strongly* H-*incomplete* iff there are wffs A and B, with one variable in each and no variables in common, such that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$. Then evidently if S is strongly H-incomplete then S is H-incomplete; the converse, however, seems to be an open question. If a system is not H-incomplete, we say it is H-*complete*. McKinsey [9] also shows that S4, S5, and all extensions of S5 (closed under substitution and detachment) are H-complete, but that there is a system between S4 and S5 which is H-incomplete. More recently Kripke [4], p. 94, has shown additionally that the modal system T and the 'Brouwersche' system B are H-complete. Åqvist [1] claims that any system between S2 and T is H-complete; his proof, however, is faulty, as is pointed out in [7]; the Corollary to Theorem 2 below gives the result.

We begin by showing that for S to be H-incomplete it is necessary and sufficient that S be the 'intersection' of two disjoint extensions, in a sense we now explain. For a system S, let T(S) be the class of theorems of S. We say that two systems S_1 and S_2 (with the same formation rules) are *disjoint* iff there are wffs A_1 and A_2 such that $\vdash_{S_1} A_1$, $\vdash_{S_2} A_2$, but not $\vdash_{S_2} A_1$ and not $\vdash_{S_1} A_2$, i.e. iff neither $T(S_1) \subseteq T(S_2)$ nor $T(S_2) \subseteq T(S_1)$. We prove:

Theorem 1. *A system* S *is* H-*incomplete iff there are disjoint systems* S_1 *and* S_2 *such that* $T(S) = T(S_1) \cap T(S_2)$.

Proof. Suppose S is H-incomplete, and that wffs A, B, with no variables in common, are such that $\vdash_S A \vee B$ yet neither $\vdash_S A$ nor $\vdash_S B$. Let S_A (S_B) be the system whose axioms are all theorems of S together with all

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