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SOME RESULTS ON FINITE AXIOMATIZABILITY IN MODAL LOGIC

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A system of propositional calculus S may be said to be *finitely axiomatizable* if there is a finite set of schemata $\{A_1, \ldots, A_n\}$ such that $\vdash_S A_i$ $(1 \leqslant i \leqslant n)$ and any theorem of S can be derived from $A_1, \ldots, A_n$ using detachment (*modus ponens*) alone. In [5], McKinsey and Tarski show, among other things, that the Lewis systems S4 and S5 are finitely axiomatizable, and in [6] the result is extended to S3[1]. The purpose of the present note is to prove a quite general theorem, due to Tarski, giving sufficient and necessary conditions under which a system is finitely axiomatizable, and to use this result to establish that the members of a certain class of modal systems, including T, S2, and E2, are not finitely axiomatizable[2].

I. In what follows, it will frequently be convenient, and never seriously ambiguous, to use the names of propositional logics as names of the corresponding classes of theorems. This is the case with the following fundamental theorem[3]. Any system is, of course, understood to be closed with respect to substitution and detachment.

Theorem 1. Let S *be any system. Then a sufficient and necessary condition for* S *to be not finitely axiomatizable is that there be an infinity of systems* $S_0, S_1, \ldots, S_n, \ldots$ *such that* $S_n \subseteq S_{n+1}$ *and* $S_n \neq S$ *for all n and* $S = \bigcup S_n$.

Proof. Suppose there are systems S_n such that $S_n \subseteq S_{n+1}$ and $S_n \neq S$ for all n, and $S = \bigcup S_n$, and consider any finite set $\{A_1, \ldots, A_m\}$ of theorem-schemata of S. Then for A_i there is a system S_{a_i} such that $\vdash_{S_{a_i}} A_i$. Let $p = \max\ \{a_1, \ldots, a_m\}$. Then $\vdash_{S_p} A_i$ for all i $(1 \leqslant i \leqslant m)$, so that any consequence of $\{A_1, \ldots, A_m\}$ is in S_p. But S_p is properly included in S, so that $\{A_1, \ldots, A_m\}$ cannot provide an axiomatization for S. Conversely, suppose S is not finitely axiomatizable, and consider an enumeration $A_1, \ldots, A_m, \ldots$ of the theorem-schemata of S. Then the systems S_n whose axiom-schemata are $\{A_1, \ldots, A_{n+1}\}$ and sole rule of inference detachment are evidently such that $S_n \subseteq S_{n+1}$ and $S = \bigcup S_n$. That $S_n \neq S$ follows from the assumption that S is not finitely axiomatizable.

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