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MATHEMATICO-PHILOSOPHICAL REMARKS ON NEW THEOREMS ANALOGOUS TO THE FUNDAMENTAL THEOREM OF ARITHMETIC

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The author dedicates this paper to the memory of Professor Thoralf Skolem; philosopher, logician, and mathematician.

Introduction: *Circa* 300 B.C., Euclid of Alexandria (*not* Euclid of Megara) borrowing partly, but not altogether, from the Pythagorean School proved (*Elements*; Book IX, Proposition 14) the following result as rendered into modern language from the Greek [1]: *If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.* This uniqueness theorem of Euclid contains the spirit if not the full essence of what is now called by many texts (see, e.g., [2], [3], and [4]) the *Fundamental Theorem of Arithmetic* (abbreviated FTA) and by nearly as many other texts (see, e.g., [5] and [6]) essentially the *Unique Factorization Theorem* (abbreviated UFT), *viz.*, Every natural number $n > 1$ has a *unique* representation of the form $n = p_1 \cdot p_2 \cdot \ldots \cdot p_k$, where k is a natural number and the p_i are primes with possible repetitions. The proof given by Euclid for his Proposition 14 of Book IX makes use of Proposition 30 of his Book VII, *viz.*, *If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers*. The modern texts cited above, among others, use this result together with formal induction in order to establish uniqueness of prime decomposition. The principal argument against Euclid having known the essence of the FTA is that throughout the *Elements* his products contain at most three factors (his argument in Book IX, Proposition 14 *holds* not only for square-free numbers with at most three factors, but for factors with repetition too; further T. L. Heath [1] in his *Scholium* to the Proposition 14 explicitly states, "In other words, a number can be resolved into prime factors in only one way."). The Greeks established their uniqueness result with the maximum generality (number of factors) that they clearly conceived with their geometrically oriented notation. Since the analogous result with *two* factors (not given in the *Elements*) is not a corollary to the result with three factors, it is reasonable to assume that formal induction either did not occur to them or else was considered logically unacceptable.

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