Notre Dame Journal of Formal Logic
Volume VI, Number 3, July 1965

AXIOMATIZATION OF PROPOSITIONAL CALCULUS WITH SHEFFER FUNCTORS[1]

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The two binary functors of the (two-valued) propositional calculus known as Sheffer functors have the property that all other functors are definable by each of them.[2] Hence, one is able to base a functionally complete propositional calculus on either of these functors, and it is of interest to axiomatize such systems, which is the main purpose of this paper. We will employ the parenthesis-free notation of Łukasiewicz, in which the Sheffer functors are given by

$Dpq = NKpq$, i.e., not both p and q
$Spq = KNpNq$, i.e., neither p nor q

It is well known that this problem has been investigated before. From the first, it was seen that it would be easier to work with D rather than with S, especially because 1) the shortest tautologies expressible with the Sheffer functors are $DDppp$ and $SSSpppSSppp$ (indicating that tautologies in S are generally longer than ones in D), and 2) the rules of detachment can be made simpler for D: with D we are allowed such rules as

D1 $\dfrac{D\alpha D\beta\gamma \quad \alpha}{\gamma}$ **D2** $\dfrac{D\alpha D\beta\gamma \quad \alpha}{\beta}$ **D3** $\dfrac{D\alpha D\beta\beta \quad \alpha}{\beta}$

while with S, the simplest rules are of the sort

S1 $\dfrac{SSS\alpha\beta\gamma\delta \quad \alpha}{\gamma}$ **S2** $\dfrac{SSS\alpha\beta\gamma SS\alpha\beta\gamma \quad \alpha}{\gamma}$ **S3** $\dfrac{SSS\alpha\alpha\beta\gamma \quad \alpha}{\beta}$

For such reasons, all investigations have been for D axioms, using the rule **D1**.[3]

Following Nicod [6], the conventional rule of detachment for D is **D1**. Clearly, this is a stronger rule than, say, **D3**, for we are allowed more freedom in the first line. However, the only investigations carried out have

Received December 16, 1964